CHAPTER 1
Introduction

Notes for the Instructor

This chapter introduces the markets for futures, forward, and options contracts and explains the activities of hedgers, speculators, and arbitrageurs. Issues concerning futures contracts such as margin requirements, settlement procedures, the role of the clearinghouse, etc are covered in Chapter 2.

Some instructors prefer to avoid any mention of options until the material on linear products in Chapters 1 to 7 has been covered. I like to introduce students to options in the first class, even though they are not mentioned again for several classes. This is because most students find options to be the most interesting of the derivatives covered and I like students to be enthusiastic about the course early on.

The way in which the material in Chapter 1 is covered is likely to depend on the backgrounds of the students. If a course in investments is a prerequisite, Chapter 1 can be regarded as a review of material already familiar to the students and can be covered fairly quickly. If an investments course is not a prerequisite, more time may be required. Increasingly some aspects of derivatives markets are being covered in introductory corporate finance courses, accounting courses, strategy courses, etc. In many instances students are, therefore, likely to have had some exposure to the material in Chapter 1. I do not require an investments elective as a prerequisite for my elective on futures and options markets and find that $1\frac{1}{2}$ to 2 hours is necessary for me to introduce the course and cover the material in Chapter 1.

To motivate students at the outset of the course, I discuss the growing importance of derivatives, how much well experts in the field are paid, etc. It is not uncommon for students who join derivatives groups, and are successful, to earn (including bonus) several hundred thousand dollars a year—or even $1 million per year—three or four years after graduating.

Towards the end of the first class I usually produce a current newspaper and describe several traded futures and options. I then ask students to guess the quoted price. Sometimes votes are taken. This is an enjoyable exercise and forces students to think actively about the nature of the contracts and the determinants of price. It usually leads to a preliminary discussion of such issues as the relationship between a futures price and the corresponding spot price, the desirability of options being exercised early, why most options sell for more than their intrinsic value, etc.

While covering the Chapter 1 material, I treat futures as the same as forwards for the purposes of discussion. I try to avoid being drawn into a discussion of such issues as the mechanics of futures, margin requirements, daily settlement procedures, and so on until I am ready. These topics are covered in Chapter 2.

As will be evident from the slides that go with this chapter, I usually introduce students to a little of the Chapter 5 material during the first class. I discuss how arbitrage arguments tie the futures price of gold to its spot price and why the futures price of a consumption commodity such as oil is not tied to its spot price in the same way. Problem 1.26 can be used to initiate the discussion.

I find that Problems 1.27 and 1.31 work well as an assignment questions. (1.31 has the advantage that it introduces students to DerivaGem early in the course.) Problem 1.28 usually generates a lively discussion. I sometimes ask students to consider it between the first and second class. We then discuss it at the beginning of the second class. Problems 1.29, 1.30, and 1.32 can be used either as assignment question or for class discussion.

QUESTIONS AND PROBLEMS

Problem 1.1.

What is the difference between a long forward position and a short forward position?

When a trader enters into a long forward contract, she is agreeing to *buy* the underlying asset for a certain price at a certain time in the future. When a trader enters into a short forward contract, she is agreeing to *sell* the underlying asset for a certain price at a certain time in the future.

Problem 1.2.

Explain carefully the difference between hedging, speculation, and arbitrage.

A trader is *hedging* when she has an exposure to the price of an asset and takes a position in a derivative to offset the exposure. In a *speculation* the trader has no exposure to offset. She is betting on the future movements in the price of the asset. *Arbitrage* involves taking a position in two or more different markets to lock in a profit.

Problem 1.3.

What is the difference between entering into a long forward contract when the forward price is \$50 and taking a long position in a call option with a strike price of \$50?

In the first case the trader is obligated to buy the asset for \$50. (The trader does not have a choice.) In the second case the trader has an option to buy the asset for \$50. (The trader does not have to exercise the option.)

Problem 1.4.

Explain carefully the difference between selling a call option and buying a put option.

Selling a call option involves giving someone else the right to buy an asset from you. It gives you a payoff of

$$-\max(S_T - K, 0) = \min(K - S_T, 0)$$

Buying a put option involves buying an option from someone else. It gives a payoff of

$$\max(K - S_T, 0)$$

In both cases the potential payoff is $K - S_T$. When you write a call option, the payoff is negative or zero. (This is because the counterparty chooses whether to exercise.) When you buy a put option, the payoff is zero or positive. (This is because you choose whether to exercise.)

Problem 1.5.

An investor enters into a short forward contract to sell 100,000 British pounds for U.S. dollars at an exchange rate of 1.9000 U.S. dollars per pound. How much does the investor gain or lose if the exchange rate at the end of the contract is (a) 1.8900 and (b) 1.9200?

(a) The investor is obligated to sell pounds for 1.9000 when they are worth 1.8900. The gain is $(1.9000 - 1.8900) \times 100,000 = \$1,000$.

(b) The investor is obligated to sell pounds for 1.9000 when they are worth 1.9200. The loss is $(1.9200 - 1.9000) \times 100,000 = \$2,000$.

Problem 1.6.

A trader enters into a short cotton futures contract when the futures price is 50 cents per pound. The contract is for the delivery of 50,000 pounds. How much does the trader gain or lose if the cotton price at the end of the contract is (a) 48.20 cents per pound; (b) 51.30 cents per pound?

(a) The trader sells for 50 cents per pound something that is worth 48.20 cents per pound. Gain $= (\$0.5000 - \$0.4820) \times 50,000 = \$900$.

(b) The trader sells for 50 cents per pound something that is worth 51.30 cents per pound. Loss $= (\$0.5130 - \$0.5000) \times 50,000 = \$650$.

Problem 1.7.

Suppose that you write a put contract with a strike price of \$40 and an expiration date in three months. The current stock price is \$41 and the contract is on 100 shares. What have you committed yourself to? How much could you gain or lose?

You have sold a put option. You have agreed to buy 100 shares for \$40 per share if the party on the other side of the contract chooses to exercise the right to sell for this price. The option will be exercised only when the price of stock is below \$40. Suppose, for example, that the option is exercised when the price is \$30. You have to buy at \$40 shares that are worth \$30; you lose \$10 per share, or \$1,000 in total. If the option is exercised when the price is \$20, you lose \$20 per share, or \$2,000 in total. The worst that can happen is that the price of the stock declines to almost zero during the three-month period. This highly unlikely event would cost you \$4,000. In return for the possible future losses, you receive the price of the option from the purchaser.

Problem 1.8.

What is the difference between the over-the-counter market and the exchange-traded market? What are the bid and offer quotes of a market maker in the over-the-counter market?

The over-the-counter market is a telephone- and computer-linked network of financial institutions, fund managers, and corporate treasurers where two participants can enter into any mutually acceptable contract. An exchange-traded market is a market organized by an exchange where traders either meet physically or communicate electronically and the contracts that can be traded have been defined by the exchange. When a market maker quotes a bid and an offer, the bid is the price at which the market maker is prepared to buy and the offer is the price at which the market maker is prepared to sell.

Problem 1.9.

You would like to speculate on a rise in the price of a certain stock. The current stock price is $29, and a three-month call with a strike of $30 costs $2.90. You have $5,800 to invest. Identify two alternative strategies, one involving an investment in the stock and the other involving investment in the option. What are the potential gains and losses from each?

One strategy would be to buy 200 shares. Another would be to buy 2,000 options. If the share price does well the second strategy will give rise to greater gains. For example, if the share price goes up to $40 you gain $[2,000 \times (\$40 - \$30)] - \$5,800 = \$14,200$ from the second strategy and only $200 \times (\$40 - \$29) = \$2,200$ from the first strategy. However, if the share price does badly, the second strategy gives greater losses. For example, if the share price goes down to $25, the first strategy leads to a loss of $200 \times (\$29 - \$25) = \$800$, whereas the second strategy leads to a loss of the whole $5,800 investment. This example shows that options contain built in leverage.

Problem 1.10.

Suppose that you own 5,000 shares worth $25 each. How can put options be used to provide you with insurance against a decline in the value of your holding over the next four months?

You could buy 5,000 put options (or 50 contracts) with a strike price of $25 and an expiration date in 4 months. This provides a type of insurance. If at the end of 4 months the stock price proves to be less than $25 you can exercise the options and sell the shares for $25 each. The cost of this strategy is the price you pay for the put options.

Problem 1.11.

When first issued, a stock provides funds for a company. Is the same true of a stock option? Discuss.

A stock option provides no funds for the company. It is a security sold by one trader to another. The company is not involved. By contrast, a stock when it is first issued is a claim sold by the company to investors and does provide funds for the company.

Problem 1.12.

Explain why a forward contract can be used for either speculation or hedging.

If a trader has an exposure to the price of an asset, she can hedge with a forward contract. If the exposure is such that the trader will gain when the price decreases and option is in these circumstances less than the price received for the option. The option will be exercised if the stock price at maturity is less than \$60.00. Note that if the stock price is between \$56.00 and \$60.00 the seller of the option makes a profit even though the option is exercised. The profit from the short position is as shown in Figure S1.2.

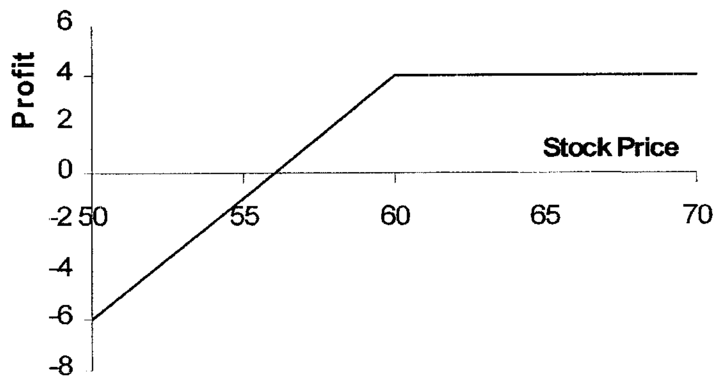


Figure S1.2 Profit from short position In Problem 1.14

Problem 1.15.

It is May and a trader writes a September call option with a strike price of \$20. The stock price is \$18, and the option price is \$2. Describe the trader's cash flows if the option is held until September and the stock price is \$25 at that time.

The trader receives an inflow of \$2 in May. Since the option is exercised, the trader also has an outflow of \$5 in September. The \$2 is the cash received from the sale of the option. The \$5 is the result of buying the stock for \$25 in September and selling it to the purchaser of the option for \$20. One contract consists of 100 options and so the cash flows for a contract are multiplied by 100.

Problem 1.16.

A trader writes a December put option with a strike price of \$30. The price of the option is \$4. Under what circumstances does the trader make a gain?

The trader makes a gain if the price of the stock is above \$26 in December. (This ignores the time value of money.)

Problem 1.17.

A company knows that it is due to receive a certain amount of a foreign currency in four months. What type of option contract is appropriate for hedging?

A long position in a four-month put option can provide insurance against the exchange rate falling below the strike price. It ensures that the foreign currency can be sold for at least the strike price.

Problem 1.18.

A United States company expects to have to pay 1 million Canadian dollars in six months. Explain how the exchange rate risk can be hedged using (a) a forward contract; (b) an option.

The company could enter into a long forward contract to buy 1 million Canadian dollars in six months. This would have the effect of locking in an exchange rate equal to the current forward exchange rate. Alternatively the company could buy a call option giving it the right (but not the obligation) to purchase 1 million Canadian dollar at a certain exchange rate in six months. This would provide insurance against a strong Canadian dollar in six months while still allowing the company to benefit from a weak Canadian dollar at that time.

Problem 1.19.

A trader enters into a short forward contract on 100 million yen. The forward exchange rate is \$0.0080 per yen. How much does the trader gain or lose if the exchange rate at the end of the contract is (a) \$0.0074 per yen; (b) \$0.0091 per yen?

(a) The trader sells 100 million yen for \$0.0080 per yen when the exchange rate is \$0.0074 per yen. The gain is 100×0.0006 millions of dollars or \$60,000.

(b) The trader sells 100 million yen for \$0.0080 per yen when the exchange rate is \$0.0091 per yen. The loss is 100×0.0011 millions of dollars or \$110,000.

Problem 1.20.

The Chicago Board of Trade offers a futures contract on long-term Treasury bonds. Characterize the traders likely to use this contract.

Most traders who use the contract will wish to do one of the following:
(a) Hedge their exposure to long-term interest rates
(b) Speculate on the future direction of long-term interest rates
(c) Arbitrage between cash and futures markets
This contract is discussed in Chapter 6.

Problem 1.21.

"Options and futures are zero-sum games." What do you think is meant by this statement?

The statement means that the gain (loss) to the party with a short position in an option is always equal to the loss (gain) to the party with the long position. The sum of the gains is zero.

Problem 1.22.

Describe the profit from the following portfolio: a long forward contract on an asset and a long European put option on the asset with the same maturity as the forward contract and a strike price that is equal to the forward price of the asset at the time the portfolio is set up.

The terminal value of the long forward contract is:

$$S_T - F_0$$

where S_T is the price of the asset at maturity and F_0 is the forward price of the asset at the time the portfolio is set up. (The delivery price in the forward contract is F_0.)

The terminal value of the put option is:

$$\max(F_0 - S_T, 0)$$

The terminal value of the portfolio is therefore

$$S_T - F_0 + \max(F_0 - S_T, 0)$$

$$= \max(0, S_T - F_0]$$

This is the same as the terminal value of a European call option with the same maturity as the forward contract and an exercise price equal to F_0. This result is illustrated in the Figure S1.3. The profit equals the terminal value less the amount paid for the option.

Problem 1.23.

In the 1980s, Bankers Trust developed index currency option notes (ICONs). These are bonds in which the amount received by the holder at maturity varies with a foreign exchange rate. One example was its trade with the Long Term Credit Bank of Japan. The ICON specified that if the yen–U.S. dollar exchange rate, S_T, is greater than 169 yen per dollar at maturity (in 1995), the holder of the bond receives \$1,000. If it is less than 169 yen per dollar, the amount received by the holder of the bond is

$$1,000 - \max\left[0,\ 1,000\left(\frac{169}{S_T} - 1\right)\right]$$

When the exchange rate is below 84.5, nothing is received by the holder at maturity. Show that this ICON is is a combination of a regular bond and two options.

Suppose that the yen exchange rate (yen per dollar) at maturity of the ICON is S_T. The payoff from the ICON is

$$\begin{array}{lll} 1,000 & \text{if} & S_T > 169 \\ 1,000 - 1,000(\frac{169}{S_T} - 1) & \text{if} & 84.5 \le S_T \le 169 \\ 0 & \text{if} & S_T < 84.5 \end{array}$$

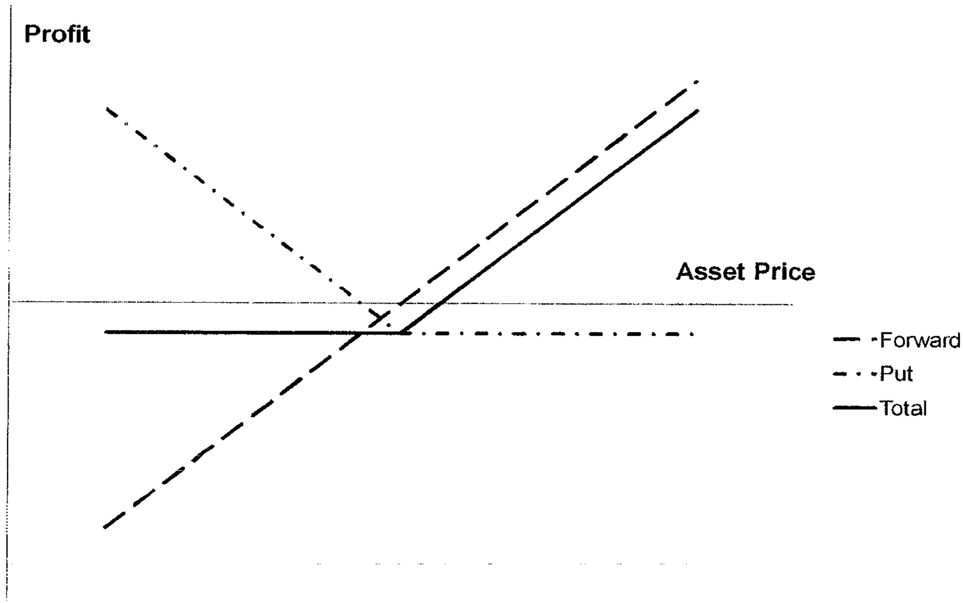


Figure S1.3 Profit from portfolio in Problem 1.22

When $84.5 \le S_T \le 169$ the payoff can be written

$$2,000 - \frac{169,000}{S_T}$$

The payoff from an ICON is the payoff from:
(a) A regular bond
(b) A short position in call options to buy 169,000 yen with an exercise price of 1/169
(c) A long position in call options to buy 169,000 yen with an exercise price of 1/84.5
This is demonstrated by the following table

	Terminal Value of Regular Bond	Terminal Value of Short Calls	Terminal Value of Long Calls	Terminal Value of Whole Position
$S_T > 169$	1,000	0	0	1000
$84.5 \le S_T \le 169$	1000	$-169,000\left(\frac{1}{S_T} - \frac{1}{169}\right)$	0	$2000 - \frac{169,000}{S_T}$
$S_T < 84.5$	1000	$-169,000\left(\frac{1}{S_T} - \frac{1}{169}\right)$	$169,000\left(\frac{1}{S_T} - \frac{1}{84.5}\right)$	0

Problem 1.24.

On July 1, 2008, a company enters into a forward contract to buy 10 million Japanese yen on January 1, 2009. On September 1, 2008, it enters into a forward contract to sell 10 million Japanese yen on January 1, 2009. Describe the payoff from this strategy.

Suppose that the forward price for the contract entered into on July 1, 2008 is F_1 and that the forward price for the contract entered into on September 1, 2008 is F_2 with both F_1 and F_2 being measured as dollars per yen. If the value of one Japanese yen (measured in U.S. dollars) is S_T on January 1, 2009, then the value of the first contract (in millions of dollars) at that time is

$$10(S_T - F_1)$$

while the value of the second contract (per yen sold) at that time is:

$$10(F_2 - S_T)$$

The total payoff from the two contracts is therefore

$$10(S_T - F_1) + 10(F_2 - S_T) = 10(F_2 - F_1)$$

Thus if the forward price for delivery on January 1, 2009 increases between July 1, 2008 and September 1, 2008 the company will make a profit.

Problem 1.25.

Suppose that USD-sterling spot and forward exchange rates are as follows:

Spot	*2.0080*
90-day forward	*2.0056*
180-day forward	*2.0018*

What opportunities are open to an arbitrageur in the following situations?

a. A 180-day European call option to buy £1 for $1.97 costs 2 cents.

b. A 90-day European put option to sell £1 for $2.04 costs 2 cents.

(a) The trader buys a 180-day call option and takes a short position in a 180-day forward contract. If S_T is the terminal spot rate, the profit from the call option is

$$\max(S_T - 1.97,\ 0) - 0.02$$

The profit from the short forward contract is

$$2.0018 - S_T$$

The profit from the strategy is therefore

$$\max(S_T - 1.97,\ 0) - 0.02 + 2.0018 - S_T$$

or

$$\max(S_T - 1.97,\ 0) + 1.9818 - S_T$$

This is

$$\begin{array}{lll} 1.9818 - S_T & \text{when} & S_T < 1.97 \\ 0.0118 & \text{when} & S_T > 1.97 \end{array}$$

This shows that the profit is always positive. The time value of money has been ignored in these calculations. However, when it is taken into account the strategy is still likely to be profitable in all circumstances. (We would require an extremely high interest rate for \$0.0118 interest to be required on an outlay of \$0.02 over a 180-day period.)

(b) The trader buys 90-day put options and takes a long position in a 90 day forward contract. If S_T is the terminal spot rate, the profit from the put option is

$$\max(2.04 - S_T, 0) - 0.020$$

The profit from the long forward contract is

$$S_T - 2.0056$$

The profit from this strategy is therefore

$$\max(2.04 - S_T, 0) - 0.020 + S_T - 2.0056$$

or

$$\max(2.04 - S_T, 0) + S_T - 2.0256$$

This is

$$\begin{array}{lll} S_T - 2.0256 & \text{when} & S_T > 2.04 \\ 0.0144 & \text{when} & S_T < 2.04 \end{array}$$

The profit is therefore always positive. Again, the time value of money has been ignored but is unlikely to affect the overall profitability of the strategy. (We would require interest rates to be extremely high for \$0.0144 interest to be required on an outlay of \$0.02 over a 90-day period.)

ASSIGNMENT QUESTIONS

Problem 1.26.

The price of gold is currently \$600 per ounce. The forward price for delivery in one year is \$800. An arbitrageur can borrow money at 10% per annum. What should the arbitrageur do? Assume that the cost of storing gold is zero and that gold provides no income.

The arbitrageur could borrow money to buy 100 ounces of gold today and short futures contracts on 100 ounces of gold for delivery in one year. This means that gold is purchased for \$600 per ounce and sold for \$800 per ounce. The return (33.3% per annum) is far greater than the 10% cost of the borrowed funds. This is such a profitable opportunity that the arbitrageur should buy as many ounces of gold as possible and short futures contracts on the same number of ounces. Unfortunately arbitrage opportunities as profitable as this rarely arise in practice.

Problem 1.27.

The current price of a stock is $94, and three-month European call options with a strike price of $95 currently sell for $4.70. An investor who feels that the price of the stock will increase is trying to decide between buying 100 shares and buying 2,000 call options (= 20 contracts). Both strategies involve an investment of $9,400. What advice would you give? How high does the stock price have to rise for the option strategy to be more profitable?

The investment in call options entails higher risks but can lead to higher returns. If the stock price stays at $94, an investor who buys call options loses $9,400 whereas an investor who buys shares neither gains nor loses anything. If the stock price rises to $120, the investor who buys call options gains

$$2000 \times (120 - 95) - 9400 = \$40,600$$

An investor who buys shares gains

$$100 \times (120 - 94) = \$2,600$$

The strategies are equally profitable if the stock price rises to a level, S, where

$$100 \times (S - 94) = 2000(S - 95) - 9400$$

or

$$S = 100$$

The option strategy is therefore more profitable if the stock price rises above $100.

Problem 1.28.

On September 12, 2006, an investor owns 100 Intel shares. As indicated in Table 1.2 the share price is $19.56 *and a January put option with a strike price $17.50 costs* $0.475. *The investor is comparing two alternatives to limit downside risk. The first is to buy one January put option contract with a strike price of $17.50. The second involves instructing a broker to sell the 100 shares as soon as Intel's price reaches 2$17.50. Discuss the advantages and disadvantages of the two strategies.*

The second alternative involves what is known as a stop or stop-loss order. It costs nothing and ensures that $1,750, or close to $1,750 is realized for the holding in the event the stock price ever falls to $17.50. The put option costs $47.50 and guarantees that the holding can be sold for $1,750 any time up to January. If the stock price falls marginally below $17.50 and then rises the option will not be exercised, but the stop-loss order will lead to the holding being liquidated. There are some circumstances where the put option alternative leads to a better outcome and some circumstances where the stop-loss order leads to a better outcome. If the stock price ends up below $17.50, the stop-loss order alternative leads to a better outcome because the cost of the option is avoided. If the stock price falls to \$17 in October and then rises to \$30 by January, the put option alternative leads to a better outcome. The investor is paying \$47.50 for the chance to benefit from this second type of outcome.

Problem 1.29.

A bond issued by Standard Oil some time ago worked as follows. The holder received no interest. At the bond's maturity the company promised to pay \$1,000 plus an additional amount based on the price of oil at that time. The additional amount was equal to the product of 170 and the excess (if any) of the price of a barrel of oil at maturity over \$25. The maximum additional amount paid was \$2,550 (which corresponds to a price of \$40 per barrel). Show that the bond is a combination of a regular bond, a long position in call options on oil with a strike price of \$25, and a short position in call options on oil with a strike price of \$40.

Suppose S_T is the price of oil at the bond's maturity. In addition to \$1000 the Standard Oil bond pays:

$$\begin{array}{ccc} S_T < \$25 & : & 0 \\ \$40 > S_T > \$25 & : & 170(S_T - 25) \\ S_T > \$40 & : & 2,550 \end{array}$$

This is the payoff from 170 call options on oil with a strike price of 25 less the payoff from 170 call options on oil with a strike price of 40. The bond is therefore equivalent to a regular bond plus a long position in 170 call options on oil with a strike price of \$25 plus a short position in 170 call options on oil with a strike price of \$40. The investor has what is termed a bull spread on oil. This is discussed in Chapter 10.

Problem 1.30.

Suppose that in the situation of Table 1.1 a corporate treasurer said: "I will have £1 million to sell in six months. If the exchange rate is less than 2.02 I want you to give me 2.02. If it is greater than 2.09 I will accept 2.09. If the exchange rate is between 2.02 and 2.09 I will sell the sterling for the exchange rate." How could you use options to satisfy the treasurer?

You sell the Treasurer a put option on GBP with a strike price of 2.02 and buy from the treasurer a call option on GBP with a strike price of 2.09. Both options are on one million pounds and have a maturity of six months. This is known as a range forward contract.

Problem 1.31.

Describe how foreign currency options can be used for hedging in the situation considered in Section 1.7 so that (a) ImportCo is guaranteed that its exchange rate will be less than 2.0700, and (b) ExportCo is guaranteed that its exchange rate will be at least 2.0400. Use DerivaGem to calculate the cost of setting up the hedge in each case assuming that the exchange rate volatility is 12%, interest rates in the United States are 5% and *interest rates in Britain are 5.7%. Assume that the current exchange rate is the average of the bid and offer in Table 1.1.*

ImportCo should buy three-month call options on £10 million with a strike price of 2.0700. ExportCo should buy three-month put options on £10 million with a strike price of 2.0400. In this case the foreign exchange rate is 2.0560 (the average of the bid and offer quotes in Table 1.1.), the (domestic) risk-free rate is 5%, the foreign risk-free rate is 5.7%, the volatility is 12%, and the time to exercise is 0.25. Using the Equity_FX_Index_Futures_Options worksheet in the DerivaGem Options Calculator select Currency as the underlying and Analytic European as the option type. The software shows that a call with a strike price of 2.07 is worth 0.0405 and a put with a strike of 2.04 is worth 0.0425. This means that the hedging would cost $0.0405 \times 10,000,000$ or \$405,000 for ImportCo and $\$0.0425 \times 10,000,000$ or \$425,000 for ExportCo.

Problem 1.32.

A trader buys a European call option and sells a European put option. The options have the same underlying asset, strike price and maturity. Describe the trader's position. Under what circumstances does the price of the call equal the price of the put?

The trader has a long European call option with strike price K and a short European put option with strike price K. Suppose the price of the underlying asset at the maturity of the option is S_T. If $S_T > K$, the call option is exercised by the investor and the put option expires worthless. The payoff from the portfolio is $S_T - K$. If $S_T < K$, the call option expires worthless and the put option is exercised against the investor. The cost to the investor is $K - S_T$. Alternatively we can say that the payoff to the investor is $S_T - K$ (a negative amount). In all cases, the payoff is $S_T - K$, the same as the payoff from the forward contract. The trader's position is equivalent to a forward contract with delivery price K.

Suppose that F is the forward price. If $K = F$, the forward contract that is created has zero value. Because the forward contract is equivalent to a long call and a short put, this shows that the price of a call equals the price of a put when the strike price is F.

CHAPTER 2
Mechanics of Futures Markets

Notes for the Instructor

This chapter explains the functioning of futures markets. I do not spend a great deal of time in class going over most of the details of how futures markets work. I let students read these for themselves. But I do find it worth spending some time going through Table 2.1 to explain the way in which margin accounts work. I also draw students' attention to the patterns of futures prices in Figure 2.2. After the essentials of the operations of futures markets have been explained, I ask students to consider Problem 2.22 in class because I find that this often reveals gaps in their understanding. I usually use about $1\frac{1}{2}$ hours to cover the material in the chapter.

There are many ways of making a discussion of futures markets fun. An easy-to-organize trading game that was explained to me by a Wall Street training manager works as follows. The instructor chooses two students to keep trading records on the front board and divides the rest of the students into about ten groups. Each group is given an identifier (e.g., A, B, C, etc) and a card with the identifier shown in big letters. They display the card when they want to make trades. The instructor chooses a seven-digit telephone number, but does not reveal this to students. The groups trade the sum of digits of the telephone number by entering long or short positions. For example, group B might bid (i.e. offer to buy) at 35. If this is accepted by another group (say group D), the record keepers show that B is long one contract at 35 and D is short one contract at 35. (If the actual sum of digits is 32, B is −3 on the trade and D is +3. The instructor controls the trading, asks for bids or offers as appropriate, and shouts trades to the record keepers. Every two minutes the instructor reveals one of the digits of the number. This game nearly always works very well for me. Trading typically starts slowly and then becomes very intense. The game gives students a sense of what futures trading is like. I insist that they use the words bid and offer rather than buy and sell.) It shows how prices are formed in markets. (After the game is over we discuss how the market price moved during the game.) The records also usually show different trading strategies. Some groups are usually speculators (all trades are long or all are short) and others are like day traders (e.g., buy at 35, sell at 36, buy at 38, sell at 39, etc). I point out to students that we need both types of traders to make the market work.

There are many stories that can be told about futures markets. Students are often interested in attempts to corner markets. I explain that the Hunt brothers' exploits in the silver market (See footnote 2 in 2.8) bankrupted them because the exchange forced them to close out their positions prior to the delivery month and as a result the price dropped. The brothers tried unsuccessfully to sue the exchange.

Business Snapshot 2.1 is an amusing story that I have often told in class. Business Snapshot 2.2 (on Long Term Capital Management) fits in well when the operation of margin accounts is being explained.

I sometimes use Problems 2.26, 2.27, and 2.28 as short hand-in assignments. Problem 2.29 is more challenging, but can be a good learning experience for students.

QUESTIONS AND PROBLEMS

Problem 2.1.

Distinguish between the terms open interest and trading volume.

The *open interest* of a futures contract at a particular time is the total number of long positions outstanding. (Equivalently, it is the total number of short positions outstanding.) The *trading volume* during a certain period of time is the number of contracts traded during this period.

Problem 2.2.

What is the difference between a local and a commission broker?

A *commission broker* trades on behalf of a client and charges a commission. A *local* trades on his or her own behalf.

Problem 2.3.

Suppose that you enter into a short futures contract to sell July silver for $10.20 per ounce on the New York Commodity Exchange. The size of the contract is 5,000 ounces. The initial margin is $4,000, and the maintenance margin is $3,000. What change in the futures price will lead to a margin call? What happens if you do not meet the margin call?

There will be a margin call when $1,000 has been lost from the margin account. This will occur when the price of silver increases by 1,000/5,000 = $0.20. The price of silver must therefore rise to $10.40 per ounce for there to be a margin call. If the margin call is not met, your broker closes out your position.

Problem 2.4.

Suppose that in September 2009 a company takes a long position in a contract on May 2010 crude oil futures. It closes out its position in March 2010. The futures price (per barrel) is $68.30 when it enters into the contract, $70.50 when it closes out its position, and $69.10 at the end of December 2009. One contract is for the delivery of 1,000 barrels. What is the company's total profit? When is it realized? How is it taxed if it is (a) a hedger and (b) a speculator? Assume that the company has a December 31 year-end.

The total profit is ($70.50 − $68.30) × 1,000 = $2,200. Of this ($69.10 − $68.30) × 1,000 = $800 is realized on a day-by-day basis between September 2009 and December 31, 2009. A further ($70.50 − $69.10) × 1,000 = $1,400 is realized on a day-by-day basis between January 1, 2009, and March 2010. A hedger would be taxed on the whole profit of $2,200 in 2010. A speculator would be taxed on $800 in 2009 and $1,400 in 2010.

Problem 2.5.

What does a stop order to sell at $2 mean? When might it be used? What does a limit order to sell at $2 mean? When might it be used?

A *stop order* to sell at $2 is an order to sell at the best available price once a price of $2 or less is reached. It could be used to limit the losses from an existing long position. A *limit order* to sell at $2 is an order to sell at a price of $2 or more. It could be used to instruct a broker that a short position should be taken, providing it can be done at a price more favorable than $2.

Problem 2.6.

What is the difference between the operation of the margin accounts administered by a clearinghouse and those administered by a broker?

The margin account administered by the clearinghouse is marked to market daily, and the clearinghouse member is required to bring the account back up to the prescribed level daily. The margin account administered by the broker is also marked to market daily. However, the account does not have to be brought up to the initial margin level on a daily basis. It has to be brought up to the initial margin level when the balance in the account falls below the maintenance margin level. The maintenance margin is usually about 75% of the initial margin.

Problem 2.7.

What differences exist in the way prices are quoted in the foreign exchange futures market, the foreign exchange spot market, and the foreign exchange forward market?

In futures markets, prices are quoted as the number of U.S. dollars per unit of foreign currency. Spot and forward rates are quoted in this way for the British pound, euro, Australian dollar, and New Zealand dollar. For other major currencies, spot and forward rates are quoted as the number of units of foreign currency per U.S. dollar.

Problem 2.8.

The party with a short position in a futures contract sometimes has options as to the precise asset that will be delivered, where delivery will take place, when delivery will take place, and so on. Do these options increase or decrease the futures price? Explain your reasoning.

These options make the contract less attractive to the party with the long position and more attractive to the party with the short position. They therefore tend to reduce the futures price.

Problem 2.9.

What are the most important aspects of the design of a new futures contract?

The most important aspects of the design of a new futures contract are the specification of the underlying asset, the size of the contract, the delivery arrangements, and the delivery months.

Problem 2.10.

Explain how margins protect investors against the possibility of default.

A margin is a sum of money deposited by an investor with his or her broker. It acts as a guarantee that the investor can cover any losses on the futures contract. The balance in the margin account is adjusted daily to reflect gains and losses on the futures contract. If losses are above a certain level, the investor is required to deposit a further margin. This system makes it unlikely that the investor will default. A similar system of margins makes it unlikely that the investor's broker will default on the contract it has with the clearinghouse member and unlikely that the clearinghouse member will default with the clearinghouse.

Problem 2.11.

A trader buys two long July futures contracts on orange juice. Each contract is for the delivery of 15,000 pounds. The current futures price is 160 cents per pound, the initial margin is \$6,000 per contract, and the maintenance margin is \$4,500 per contract. What price change would lead to a margin call? Under what circumstances could \$2,000 be withdrawn from the margin account?

There is a margin call if \$1,500 is lost on one contract. This happens if the futures price of orange juice falls by 10 cents to 150 cents per lb. \$2,000 can be withdrawn from the margin account if there is a gain on one contract of \$1,000. This will happen if the futures price rises by 6.67 cents to 166.67 cents per lb.

Problem 2.12.

Show that if the futures price of a commodity is greater than the spot price during the delivery period there is an arbitrage opportunity. Does an arbitrage opportunity exist if the futures price is less than the spot price? Explain your answer.

If the futures price is greater than the spot price during the delivery period, an arbitrageur buys the asset, shorts a futures contract, and makes delivery for an immediate profit. If the futures price is less than the spot price during the delivery period, there is no similar perfect arbitrage strategy. An arbitrageur can take a long futures position but cannot force immediate delivery of the asset. The decision on when delivery will be made is made by the party with the short position. Nevertheless companies interested in acquiring the asset will find it attractive to enter into a long futures contract and wait for delivery to be made.

Problem 2.13.

Explain the difference between a market-if-touched order and a stop order.

A market-if-touched order is executed at the best available price after a trade occurs at a specified price or at a price more favorable than the specified price. A stop order is executed at the best available price after there is a bid or offer at the specified price or at a price less favorable than the specified price.

Problem 2.14.

Explain what a stop-limit order to sell at 20.30 with a limit of 20.10 means.

A stop-limit order to sell at 20.30 with a limit of 20.10 means that as soon as there is a bid at 20.30 the contract should be sold providing this can be done at 20.10 or a higher price.

Problem 2.15.

At the end of one day a clearinghouse member is long 100 contracts, and the settlement price is \$50,000 per contract. The original margin is \$2,000 per contract. On the following day the member becomes responsible for clearing an additional 20 long contracts, entered into at a price of \$51,000 per contract. The settlement price at the end of this day is \$50,200. How much does the member have to add to its margin account with the exchange clearinghouse?

The clearinghouse member is required to provide $20 \times \$2,000 = \$40,000$ as initial margin for the new contracts. There is a gain of $(50{,}200 - 50{,}000) \times 100 = \$20{,}000$ on the existing contracts. There is also a loss of $(51,000 - 50,200) \times 20 = \$16,000$ on the new contracts. The member must therefore add

$$40,000 - 20,000 + 16,000 = \$36,000$$

to the margin account.

Problem 2.16.

On July 1, 2009, a Japanese company enters into a forward contract to buy \$1 million on January 1, 2010. On September 1, 2009, it enters into a forward contract to sell \$1 million on January 1, 2010. Describe the profit or loss the company will make in yen as a function of the forward exchange rates on July 1, 2009 and September 1, 2009.

Suppose F_1 and F_2 are the forward exchange rates for the contracts entered into July 1, 2009 and September 1, 2009, and S is the spot rate on January 1, 2010. (All exchange rates are measured as yen per dollar). The payoff from the first contract is $(S - F_1)$ million yen and the payoff from the second contract is $(F_2 - S)$ million yen. The total payoff is therefore $(S - F_1) + (F_2 - S) = (F_2 - F_1)$ million yen.

Problem 2.17.

The forward price on the Swiss franc for delivery in 45 days is quoted as 1.2500. The futures price for a contract that will be delivered in 45 days is 0.7980. Explain these two quotes. Which is more favorable for an investor wanting to sell Swiss francs?

The 1.2500 forward quote is the number of Swiss francs per dollar. The 0.7980 futures quote is the number of dollars per Swiss franc. When quoted in the same way as the futures price the forward price is $1/1.2500 = 0.8000$. The Swiss franc is therefore more valuable in the forward market than in the futures market. The forward market is therefore more attractive for an investor wanting to sell Swiss francs.

Problem 2.18.

Suppose you call your broker and issue instructions to sell one July hogs contract. Describe what happens.

Hog futures are traded on the Chicago Mercantile Exchange. (See Table 2.2). The broker will request some initial margin. The order will be relayed by telephone to your broker's trading desk on the floor of the exchange (or to the trading desk of another broker).

It will be sent by messenger to a commission broker who will execute the trade according to your instructions. Confirmation of the trade eventually reaches you. If there are adverse movements in the futures price your broker may contact you to request additional margin.

Problem 2.19.

"Speculation in futures markets is pure gambling. It is not in the public interest to allow speculators to trade on a futures exchange." Discuss this viewpoint.

Speculators are important market participants because they add liquidity to the market. However, contracts must be useful for hedging as well as speculation. This is because regulators generally only approve contracts when they are likely to be of interest to hedgers as well as speculators.

Problem 2.20.

Identify the contracts that have the highest open interest in Table 2.2.

The table does not show contracts for all maturities. In the Metals and Petroleum category it appears that crude oil has the highest open interest. In the agricultural category it appears that corn has the highest open interest.

Problem 2.21.

What do you think would happen if an exchange started trading a contract in which the quality of the underlying asset was incompletely specified?

The contract would not be a success. Parties with short positions would hold their contracts until delivery and then deliver the cheapest form of the asset. This might well be viewed by the party with the long position as garbage! Once news of the quality problem became widely known no one would be prepared to buy the contract. This shows that futures contracts are feasible only when there are rigorous standards within an industry for defining the quality of the asset. Many futures contracts have in practice failed because of the problem of defining quality.

Problem 2.22.

"When a futures contract is traded on the floor of the exchange, it may be the case that the open interest increases by one, stays the same, or decreases by one." Explain this statement.

If both sides of the transaction are entering into a new contract, the open interest increases by one. If both sides of the transaction are closing out existing positions, the open interest decreases by one. If one party is entering into a new contract while the other party is closing out an existing position, the open interest stays the same.

Problem 2.23.

Suppose that on October 24, 2009, a company sells one April 2010 live-cattle futures contract. It closes out its position on January 21, 2010. The futures price (per pound) is 91.20 cents when it enters into the contract, 88.30 cents when it close out its position, and 88.80 cents at the end of December 2009. One contract is for the delivery of 40,000 pounds of cattle. What is the total profit? How is it taxed if the company is (a) a hedger and (b) a speculator? Assume that the company has a December 31 year end.

The total profit is

$$40,000 \times (0.9120 - 0.8830) = \$1,160$$

If you are a hedger this is all taxed in 2009. If you are a speculator

$$40,000 \times (0.9120 - 0.8880) = \$960$$

is taxed in 2009 and

$$40,000 \times (0.8880 - 0.8830) = \$200$$

is taxed in 2010.

Problem 2.24.

A cattle farmer expects to have 120,000 pounds of live cattle to sell in three months. The live-cattle futures contract on the Chicago Mercantile Exchange is for the delivery of 40,000 pounds of cattle. How can the farmer use the contract for hedging? From the farmer's viewpoint, what are the pros and cons of hedging?

The farmer can short 3 contracts that have 3 months to maturity. If the price of cattle falls, the gain on the futures contract will offset the loss on the sale of the cattle. If the price of cattle rises, the gain on the sale of the cattle will be offset by the loss on the futures contract. Using futures contracts to hedge has the advantage that it can at no cost reduce risk to almost zero. Its disadvantage is that the farmer no longer gains from favorable movements in cattle prices.

Problem 2.25.

It is now July 2008. A mining company has just discovered a small deposit of gold. It will take six months to construct the mine. The gold will then be extracted on a more or less continuous basis for one year. Futures contracts on gold are available on the New York Commodity Exchange. There are delivery months every two months from August 2008 to December 2009. Each contract is for the delivery of 100 ounces. Discuss how the mining company might use futures markets for hedging.

The mining company can estimate its production on a month by month basis. It can then short futures contracts to lock in the price received for the gold. For example, if a total of 3,000 ounces are expected to be produced in January 2009 and February 2009, the price received for this production can be hedged by shorting a total of 30 February 2009 contracts.

ASSIGNMENT QUESTIONS

Problem 2.26.

A company enters into a short futures contract to sell 5,000 bushels of wheat for 450 cents per bushel. The initial margin is $3,000 and the maintenance margin is $2,000. What price change would lead to a margin call? Under what circumstances could $1,500 be withdrawn from the margin account?

There is a margin call if $1000 is lost on the contract. This will happen if the price of wheat futures rises by 20 cents from 450 cents to 470 cents per bushel. $1500 can be withdrawn if the futures price falls by 30 cents to 420 cents per bushel.

Problem 2.27.

Suppose that there are no storage costs for crude oil and the interest rate for borrowing or lending is 5% per annum. How could you make money on January 8, 2007 by trading June 2007 and December 2007 contracts on crude oil? Use Table 2.2.

The June 2007 settlement price for oil is $60.01 per barrel. The December 2007 settlement price for oil is $62.94 per barrel. You could go long one June 2007 oil contract and short one December 2007 contract. In June 2007 you take delivery of the oil borrowing $60.01 per barrel at 5% to meet cash outflows. The interest accumulated in six months is about $60.01 \times 0.05 \times 0.5$ or $1.50. In December the oil is sold for $62.94 and $60.01 + 1.50 =$ $61.51 per barrel has to be repaid on the loan. The strategy therefore leads to a profit of $62.94 - 61.51$ or $1.43 per barrel. Note that this profit is independent of the actual price of oil in June 2007 or December 2007. It will be slightly affected by the daily settlement procedures.

Problem 2.28.

What position is equivalent to a long forward contract to buy an asset at K on a certain date and a put option to sell it for K on that date.

Suppose an investor has a long European call option with strike price K and a short European put option with strike price K. Suppose the price of the underlying asset at the maturity of the option is S_T. If $S_T > K$, the call option is exercised by the investor and the put option expires worthless. The payoff from the portfolio is $S_T - K$. If $S_T < K$, the call option expires worthless and the put option is exercised against the investor. The cost to the investor is $K - S_T$. Alternatively we can say that the payoff to the investor is $S_T - K$ (a negative amount). In all cases, the payoff is $S_T - K$, the same as the payoff from the forward contract.

Suppose that F is the forward price. If $K = F$, the forward contract that is created has zero value. Because the forward contract is equivalent to a long call and a short put, this shows that the price of a call equals the price of a put when the strike price is F.

Problem 2.29.

The author's Web page (www.rotman.utoronto.ca/~hull/data) contains daily closing prices for crude oil futures and gold futures contracts. (Both contracts are traded on NYMEX.) You are required to download the data and answer the following:

a. *How high do the maintenance margin levels for oil and gold have to be set so that there is a 1% chance that an investor with a balance slightly above the maintenence margin level on a particular day has a negative balance two days later? How high do they have to be for a 0.1% chance? Assume daily price changes are normally distributed with mean zero. Explain why NYMEX might be interested in this calculation.*

b. *Imagine an investor who starts with a long position in the oil contract at the beginning of the period covered by the data and keeps the contract for the whole of the period of time covered by the data. Margin balances in excess of the initial margin are withdrawn. Use the maintenance margin you calculated in part (a) for a 1% risk level and assume that the maintenance margin is 75% of the initial margin. Calculate the number of margin calls and the number of times the investor has a negative margin balance. Assume that all margin calls are met in your calculations. Repeat the calculations for an investor who starts with a short position in the gold contract.*

(a) For gold the standard deviation of daily changes is \$2.77 per ounce or \$277 per contract. For a 1% risk this means that the maintenance margin should be set at $277 \times \sqrt{2} \times 2.33 = 912$. For a 0.1% risk the maintenance margin should be set at $277 \times \sqrt{2} \times 3.09 = 1,210$.
For crude oil the standard deviation of daily changes is \$0.31 per barrel or \$310 per contract. For a 1% risk this means that the maintenance margin should be set at $310 \times \sqrt{2} \times 2.33 = 1,021$. For a 0.1% risk the maintenance margin should be set at $310 \times \sqrt{2} \times 3.09 = 1,355$.
NYMEX is interested in these types of calculations because it wants to set the maintenance margin level so that the balance in a trader's margin account has a very low probability of becoming negative. If a trader started with a balance just above the maintenance margin level and the market moved against her, there would be a margin call at the end of the first day and the trader would have until the end of the second day to meet the margin call. It is therefore the possibility of a large futures price movement over a two-day period that is of concern to NYMEX.

(b) The initial margin is set at 1,362 for crude oil. (This is the maintenance margin divided by 0.75.) There are 151 margin calls and 7 times (out of 1201 days) where the investor is tempted to walk away. The initial margin is set at 1,215 for gold. There are 111 margin calls and 3 times (out of 826 days) when the investor is tempted to walk away. When the 0.1% risk level is used there are 3 times when the oil investor might walk away and 6 times when the gold investor might do so. These results suggest that

extreme movements occur more often that the normal distribution would suggest.
Here are some notes on how I handled the Excel calculations. Suppose that the initial margin is in cell Q1 and the maintenance margin is in cell Q2. Suppose further that the change in the oil futures price is in column D of the spreadsheet and the margin balance is in column E. Consider cell E7. This is updated with an instruction of the form:

$$= \mathrm{IF}(\mathrm{E6} < \$\mathrm{Q}\$2, \$\mathrm{Q}\$1, \mathrm{IF}(\mathrm{E6} + \mathrm{D7} * 1000 > \$\mathrm{Q}\$1, \$\mathrm{Q}\$1, \mathrm{E6} + \mathrm{D7} * 1000))$$

Returning 1 in cell F7 if there has been a margin call and zero otherwise requires an instruction of the form:

$$= \mathrm{IF}(\mathrm{E7} < \$\mathrm{Q}\$2, 1, 0)$$

Returning 1 in cell G7 if there has been an incentive to walk away and zero otherwise requires an instruction of the form:

$$= \mathrm{IF}(\mathrm{E6} + \mathrm{D7} * 1000 < 0, 1, 0)$$

CHAPTER 3

Hedging Strategies Using Futures

Notes for the Instructor

This chapter discusses how long and short futures positions are used for hedging. It covers basis risk, hedge ratios, the use of stock index futures, and how to roll a hedge forward.

A number of people have pointed out a small inconsistency between the material in Chapter 3 and the CFA material in the previous edition. The issue is whether you base the number of contracts used for hedging on the futures price of the assets underlying a futures contract or the spot price of these assets. To be consistent with CFA, this edition does the former. The argument for doing so is that it is a way of adjusting for the marking to market of futures contracts. (See "tailing the hedge" material on page 58 and Problem 5.23 in Chapter 5.)

As will be evident from the slides, I cover the material in the chapter in the order in which it is presented. The section on arguments for and against hedging often generates a lively discussion. It is important to emphasize that the purpose of hedging is to reduce the standard deviation of the outcome, not to increase its expected value. I usually discuss Problem 3.17 at some stage to emphasize the point that, even in relatively simple situations, it is easy to make incorrect hedging decisions when you do not look at the big picture.

Business Snapshot 3.1 discusses hedging by gold mining companies. I use this to emphasize the importance of communicating with shareholders. I also like to discuss how investment banks hedge their risks when they enter into forward contracts with gold producers. (This is the second part of Business Snapshot 3.1.) I also like to ask students about the determinants of gold lease rates. If more gold producers choose to hedge, does the gold lease rate go up or down? (The answer is that it goes up because there is a greater demand on the part of investment banks for gold borrowing.)

Any of the Problems 3.23 to 3.26 can be used as assignment questions. My favorite is Problem 3.26.

QUESTIONS AND PROBLEMS

Problem 3.1.

Under what circumstances are (a) a short hedge and (b) a long hedge appropriate?

A *short hedge* is appropriate when a company owns an asset and expects to sell that asset in the future. It can also be used when the company does not currently own the asset but expects to do so at some time in the future. A *long hedge* is appropriate when a company knows it will have to purchase an asset in the future. It can also be used to offset the risk from an existing short position.

Problem 3.2.

Explain what is meant by basis risk when futures contracts are used for hedging.

Basis risk arises from the hedger's uncertainty as to the difference between the spot price and futures price at the expiration of the hedge.

Problem 3.3.

Explain what is meant by a perfect hedge. Does a perfect hedge always lead to a better outcome than an imperfect hedge? Explain your answer.

A *perfect hedge* is one that completely eliminates the hedger's risk. A perfect hedge does not always lead to a better outcome than an imperfect hedge. It just leads to a more certain outcome. Consider a company that hedges its exposure to the price of an asset. Suppose the asset's price movements prove to be favorable to the company. A perfect hedge totally neutralizes the company's gain from these favorable price movements. An imperfect hedge, which only partially neutralizes the gains, might well give a better outcome.

Problem 3.4.

Under what circumstances does a minimum-variance hedge portfolio lead to no hedging at all?

A minimum variance hedge leads to no hedging when the coefficient of correlation between the futures price changes and changes in the price of the asset being hedged is zero.

Problem 3.5.

Give three reasons that the treasurer of a company might not hedge the company's exposure to a particular risk.

(a) If the company's competitors are not hedging, the treasurer might feel that the company will experience less risk if it does not hedge. (See Table 3.1.) (b) The shareholders might not want the company to hedge. (c) If there is a loss on the hedge and a gain from the company's exposure to the underlying asset, the treasurer might feel that he or she will have difficulty justifying the hedging to other executives within the organization.

Problem 3.6.

Suppose that the standard deviation of quarterly changes in the prices of a commodity is \$0.65, the standard deviation of quarterly changes in a futures price on the commodity is \$0.81, and the coefficient of correlation between the two changes is 0.8. What is the optimal hedge ratio for a three-month contract? What does it mean?

The optimal hedge ratio is

$$0.8 \times \frac{0.65}{0.81} = 0.642$$

This means that the size of the futures position should be 64.2% of the size of the company's exposure in a three-month hedge.

Problem 3.7.

A company has a \$20 million portfolio with a beta of 1.2. It would like to use futures contracts on the S&P 500 to hedge its risk. The index futures is currently standing at 1080, and each contract is for delivery of \$250 times the index. What is the hedge that minimizes risk? What should the company do if it wants to reduce the beta of the portfolio to 0.6?

The formula for the number of contracts that should be shorted gives

$$1.2 \times \frac{20,000,000}{1080 \times 250} = 88.9$$

Rounding to the nearest whole number, 89 contracts should be shorted. To reduce the beta to 0.6, half of this position, or a short position in 44 contracts, is required.

Problem 3.8.

In the Chicago Board of Trade's corn futures contract, the following delivery months are available: March, May, July, September, and December. State the contract that should be used for hedging when the expiration of the hedge is in

a. *June*
b. *July*
c. *January*

A good rule of thumb is to choose a futures contract that has a delivery month as close as possible to, but later than, the month containing the expiration of the hedge. The contracts that should be used are therefore (a) July, (b) September, and (c) March.

Problem 3.9.

Does a perfect hedge always succeed in locking in the current spot price of an asset for a future transaction? Explain your answer.

No. Consider, for example, the use of a forward contract to hedge a known cash inflow in a foreign currency. The forward contract locks in the forward exchange rate — which is in general different from the spot exchange rate.

Problem 3.10.

Explain why a short hedger's position improves when the basis strengthens unexpectedly and worsens when the basis weakens unexpectedly.

The basis is the amount by which the spot price exceeds the futures price. A short hedger is long the asset and short futures contracts. The value of his or her position therefore improves as the basis increases. Similarly it worsens as the basis decreases.

Problem 3.11.

Imagine you are the treasurer of a Japanese company exporting electronic equipment to the United States. Discuss how you would design a foreign exchange hedging strategy and the arguments you would use to sell the strategy to your fellow executives.

The simple answer to this question is that the treasurer should (a) estimate the company's future cash flows in Japanese yen and U.S. dollars and (b) enter into forward and futures contracts to lock in the exchange rate for the U.S. dollar cash flows.

However, this is not the whole story. As the gold jewelry example in Table 3.1 shows, the company should examine whether the magnitudes of the foreign cash flows depend on the exchange rate. For example, will the company be able to raise the price of its product in U.S. dollars if the yen appreciates? If the company can do so, its foreign exchange exposure may be quite low. The key estimates required are those showing the overall effect on the company's profitability of changes in the exchange rate at various times in the future. Once these estimates have been produced the company can choose between using futures and options to hedge its risk. The results of the analysis should be presented carefully to other executives. It should be explained that a hedge does not ensure that profits will be higher. It means that profit will be more certain. When futures/forwards are used both the downside and upside are eliminated. With options a premium is paid to eliminate only the downside.

Problem 3.12.

Suppose that in Example 3.2 of Section 3.3 the company decides to use a hedge ratio of 0.8. How does the decision affect the way in which the hedge is implemented and the result?

If the hedge ratio is 0.8, the company takes a long position in 16 NYM December oil futures contracts on June 8 when the futures price is \$68.00. It closes out its position on November 10. The spot price and futures price at this time are \$70.00 and \$69.10. The gain on the futures position is

$$(69.10 - 68.00) \times 16{,}000 = 17{,}600$$

The effective cost of the oil is therefore

$$20{,}000 \times 70 - 17{,}600 = 1{,}382{,}400$$

or \$69.12 per barrel. (This compares with \$68.90 per barrel when the company is fully hedged.)

Problem 3.13.

"If the minimum-variance hedge ratio is calculated as 1.0, the hedge must be perfect." Is this statement true? Explain your answer.

The statement is not true. The minimum variance hedge ratio is

$$\rho\frac{\sigma_S}{\sigma_F}$$

It is 1.0 when $\rho = 0.5$ and $\sigma_S = 2\sigma_F$. Since $\rho < 1.0$ the hedge is clearly not perfect.

Problem 3.14.

"If there is no basis risk, the minimum variance hedge ratio is always 1.0." Is this statement true? Explain your answer.

The statement is true. Using the notation in the text, if the hedge ratio is 1.0, the hedger locks in a price of $F_1 + b_2$. Since both F_1 and b_2 are known this has a variance of zero and must be the best hedge.

Problem 3.15.

"For an asset where futures prices are usually less than spot prices, long hedges are likely to be particularly attractive." Explain this statement.

A company that knows it will purchase a commodity in the future is able to lock in a price close to the futures price. This is likely to be particularly attractive when the futures price is less than the spot price.

Problem 3.16.

The standard deviation of monthly changes in the spot price of live cattle is (in cents per pound) 1.2. The standard deviation of monthly changes in the futures price of live cattle for the closest contract is 1.4. The correlation between the futures price changes and the spot price changes is 0.7. It is now October 15. A beef producer is committed to purchasing 200,000 pounds of live cattle on November 15. The producer wants to use the December live-cattle futures contracts to hedge its risk. Each contract is for the delivery of 40,000 pounds of cattle. What strategy should the beef producer follow?

The optimal hedge ratio is

$$0.7 \times \frac{1.2}{1.4} = 0.6$$

The beef producer requires a long position in $200000 \times 0.6 = 120,000$ lbs of cattle. The beef producer should therefore take a long position in 3 December contracts closing out the position on November 15.

Problem 3.17.

A corn farmer argues "I do not use futures contracts for hedging. My real risk is not the price of corn. It is that my whole crop gets wiped out by the weather."Discuss this viewpoint. Should the farmer estimate his or her expected production of corn and hedge to try to lock in a price for expected production?

Suppose that the weather is bad and the farmer's production is lower than expected. Other farmers are likely to have been affected similarly. Corn production overall will be low and as a consequence the price of corn will be relatively high. The farmer is likely to be overhedged relative to actual production. The farmer's problems arising from the bad harvest will be made worse by losses on the short futures position. This problem emphasizes the importance of looking at the big picture when hedging. The farmer is correct to question whether hedging price risk while ignoring other risks is a good strategy.

Problem 3.18.

On July 1, an investor holds 50,000 shares of a certain stock. The market price is $30 per share. The investor is interested in hedging against movements in the market over the next month and decides to use the September Mini S&P 500 futures contract. The index futures price is currently 1,500 and one contract is for delivery of $50 times the index. The beta of the stock is 1.3. What strategy should the investor follow?

A short position in

$$1.3 \times \frac{50,000 \times 30}{50 \times 1,500} = 26$$

contracts is required.

Problem 3.19.

Suppose that in Table 3.5 the company decides to use a hedge ratio of 1.5. How does the decision affect the way the hedge is implemented and the result?

If the company uses a hedge ratio of 1.5 in Table 3.5 it would at each stage short 150 contracts. The gain from the futures contracts would be

$$1.50 \times 1.70 = \$2.55 \text{ per barrel}$$

and the company would be $0.85 per barrel better off.

Problem 3.20.

A futures contract is used for hedging. Explain why the marking to market of the contract can give rise to cash flow problems.

Suppose that you enter into a short futures contract to hedge the sale of a asset in six months. If the price of the asset rises sharply during the six months, the futures price will also rise and you may get margin calls. The margin calls will lead to cash outflows. Eventually the cash outflows will be offset by the extra amount you get when you sell the asset, but there is a mismatch in the timing of the cash outflows and inflows. Your cash outflows occur earlier than your cash inflows. A similar situation could arise if you used a long position in a futures contract to hedge the purchase of an asset and the asset's price fell sharply. An extreme example of what we are talking about here is provided by Metallgesellschaft (see Business Snapshot 3.2).

Problem 3.21.

An airline executive has argued: "There is no point in our using oil futures. There is just as much chance that the price of oil in the future will be less than the futures price as there is that it will be greater than this price." Discuss the executive's viewpoint.

It may well be true that there is just as much chance that the price of oil in the future will be above the futures price as that it will be below the futures price. This means that the use of a futures contract for speculation would be like betting on whether a coin comes up heads or tails. But it might make sense for the airline to use futures for hedging rather than speculation. The futures contract then has the effect of reducing risks. It can be argued that an airline should not expose its shareholders to risks associated with the future price of oil when there are contracts available to hedge the risks.

Problem 3.22.

Suppose the one-year gold lease rate is 1.5% and the one-year risk-free rate is 5.0%. Both rates are compounded annually. Use the discussion in Business Snapshot 3.1 to calculate the maximum one-year forward price Goldman Sachs should quote for gold when the spot price is $600.

Goldman Sachs can borrow 1 ounce of gold and sell it for $600. It invests the $600 at 5% so that it becomes $630 at the end of the year. It must pay the lease rate of 1.5% on $600. This is $9 and leaves it with $621. It follows that if it agrees to buy the gold for less than $621 in one year it will make a profit.

ASSIGNMENT QUESTIONS

Problem 3.23.

The following table gives data on monthly changes in the spot price and the futures price for a certain commodity. Use the data to calculate a minimum variance hedge ratio.

Spot Price Change	+0.50	+0.61	−0.22	−0.35	+0.79
Futures Price Change	+0.56	+0.63	−0.12	−0.44	+0.60
Spot Price Change	+0.04	+0.15	+0.70	−0.51	−0.41
Futures price change	−0.06	+0.01	+0.80	−0.56	−0.46

Denote x_i and y_i by the i-th observation on the change in the futures price and the change in the spot price respectively.

$$\sum x_i = 0.96 \quad \sum y_i = 1.30$$

$$\sum x_i^2 = 2.4474 \quad \sum y_i^2 = 2.3594$$

$$\sum x_i y_i = 2.352$$

An estimate of σ_F is

$$\sqrt{\frac{2.4474}{9} - \frac{0.96^2}{10 \times 9}} = 0.5116$$

An estimate of σ_S is

$$\sqrt{\frac{2.3594}{9} - \frac{1.30^2}{10 \times 9}} = 0.4933$$

An estimate of ρ is

$$\frac{10 \times 2.352 - 0.96 \times 1.30}{\sqrt{(10 \times 2.4474 - 0.96^2)(10 \times 2.3594 - 1.30^2)}} = 0.981$$

The minimum variance hedge ratio is

$$\rho \frac{\sigma_S}{\sigma_F} = 0.981 \times \frac{0.4933}{0.5116} = 0.946$$

Problem 3.24.

It is July 16. A company has a portfolio of stocks worth \$100 million. The beta of the portfolio is 1.2. The company would like to use the CME December futures contract on the S&P 500 to change the beta of the portfolio to 0.5 during the period July 16 to November 16. The index futures price is currently 1,000, and each contract is on \$250 times the index.

a. *What position should the company take?*
b. *Suppose that the company changes its mind and decides to increase the beta of the portfolio from 1.2 to 1.5. What position in futures contracts should it take?*

(a) The company should short

$$\frac{(1.2 - 0.5) \times 100,000,000}{1000 \times 250}$$

or 280 contracts.

(b) The company should take a long position in

$$\frac{(1.5 - 1.2) \times 100,000,000}{1000 \times 250}$$

or 120 contracts.

Problem 3.25.

A fund manager has a portfolio worth \$50 million with a beta of 0.87. The manager is concerned about the performance of the market over the next two months and plans to use three-month futures contracts on the S&P 500 to hedge the risk. The current level of the index is 1250, one contract is on 250 times the index, the risk-free rate is 6% per annum, and the dividend yield on the index is 3% per annum. The current 3 month futures price is 1259.

a. *What position should the fund manager take to eliminate all exposure to the market over the next two months?*
b. *Calculate the effect of your strategy on the fund manager's returns if the level of the market in two months is 1,000, 1,100, 1,200, 1,300, and 1,400. Assume that the one-month futures price is 0.25% higher than the index level at this time.*

(a) The number of contracts the fund manager should short is

$$0.87 \times \frac{50,000,000}{1259 \times 250} = 138.20$$

Rounding to the nearest whole number, 138 contracts should be shorted.

(b) The following table shows that the strategy has the effect of locking in a return of close to \$490,000. To illustrate the calculations in the table consider the first column. If the index in two months is 1,000, the futures price is $1000 \times 1.0025 = 1002.50$ The gain on the short futures position is therefore

$$(1259 - 1002.50) \times 250 \times 138 = \$8,849,250$$

The return on the index is $3 \times 2/12$=0.5% in the form of dividend and $-250/1250 = -20\%$ in the form of capital gains. The total return on the index is therefore −19.5%. The risk-free rate is 1% per two months. The return is therefore −20.5% in excess of the risk-free rate. From the capital asset pricing model we expect the return on the portfolio to be $0.87 \times -20.5\% = -17.835\%$ in excess of the risk-free rate. The portfolio return is therefore −16.835%. The loss on the portfolio is $0.16835 \times 50,000,000$ or \$8,417,500. When this is combined with the gain on the futures the total gain is \$431,750.

Index in Two months	1000	1100	1200	1300	1400
Futures Price (\$)	1002.50	1102.75	1203.00	1303.25	1403.50
Gain on Futures (\$)	8,849,250	5,390,625	1,932,000	-1,526,625	-4,985,250
Index Return	-19.5%	-11.5%	-3.5%	4.5%	12.5%
Excess Ind. Return	-20.5%	-12.5%	-4.5%	3.5%	11.5%
Excess Port. Return	-17.835%	-10.875%	-3.915%	3.045%	10.005%
Port. Return	-16.835%	-9.875%	-2.915%	4.045%	11.005%
Port. Gain (\$)	-8,417,500	-4,937,500	-1,457,500	2,022,500	5,502,500
Total Gain (\$)	431,750	453,125	488,500	495,875	517,250

Problem 3.26.

It is now October 2007. A company anticipates that it will purchase 1 million pounds of copper in each of February 2008, August 2008, February 2009, and August 2009. The company has decided to use the futures contracts traded in the COMEX division of the New York Mercantile Exchange to hedge its risk. One contract is for the delivery of 25,000 pounds of copper. The initial margin is \$2,000 per contract and the maintenance margin is \$1,500 per contract. The company's policy is to hedge 80% of its exposure. Contracts with maturities up to 13 months into the future are considered to have sufficient liquidity to meet the company's needs. Devise a hedging strategy for the company. Do not make the tailing adjustments described in Section 3.4.

Assume the market prices (in cents per pound) today and at future dates are as follows. What is the impact of the strategy you propose on the price the company pays for copper? What is the initial margin requirement in October 2004? Is the company subject to any margin calls?

Date	*Oct 2007*	*Feb 2008*	*Aug 2008*	*Feb 2009*	*Aug 2009*
Spot Price	*372.00*	*369.00*	*365.00*	*377.00*	*388.00*
Mar 2008 Futures Price	*372.30*	*369.10*			
Sep 2008 Futures Price	*372.80*	*370.20*	*364.80*		
Mar 2009 Futures Price		*370.70*	*364.30*	*376.70*	
Sep 2009 Futures Price			*364.20*	*376.50*	*388.20*

What is the impact of the strategy you propose on the price the company pays for copper? What is the initial margin requirement in October 2007? Is the company subject to any margin calls?

To hedge the February 2008 purchase the company should take a long position in March 2008 contracts for the delivery of 800,000 pounds of copper. The total number of contracts required is $800,000/25,000 = 32$. Similarly a long position in 32 September 2008 contracts is required to hedge the August 2008 purchase. For the February 2009 purchase the company could take a long position in 32 September 2008 contracts and roll them into March 2009 contracts during August 2008. (As an alternative, the company could hedge the February 2009 purchase by taking a long position in 32 March 2008 contracts and rolling them into March 2009 contracts.) For the August 2009 purchase the company could take a long position in 32 September 2008 and roll them into September 2009 contracts during August 2008.

The strategy is therefore as follows

Oct. 2007: Enter into long position in 96 Sept. 2008 contracts
Enter into a long position in 32 Mar. 2008 contracts
Feb. 2008: Close out 32 Mar. 2008 contracts
Aug. 2008: Close out 96 Sept. 2008 contracts
Enter into long position in 32 Mar. 2009 contracts
Enter into long position in 32 Sept. 2009 contracts
Feb. 2009: Close out 32 Mar. 2009 contracts
Aug. 2009: Close out 32 Sept. 2009 contracts

With the market prices shown the company pays

$$369.00 + 0.8 \times (372.30 - 369.10) = 371.56$$

for copper in February, 2008. It pays

$$365.00 + 0.8 \times (372.80 - 364.80) = 371.40$$

for copper in August 2008. As far as the February 2009 purchase is concerned, it loses $372.80 - 364.80 = 8.00$ on the September 2008 futures and gains $376.70 - 364.30 = 12.40$

on the February 2009 futures. The net price paid is therefore

$$377.00 + 0.8 \times 8.00 - 0.8 \times 12.40 = 373.48$$

As far as the August 2009 purchase is concerned, it loses $372.80 - 364.80 = 8.00$ on the September 2008 futures and gains $388.20 - 364.20 = 24.00$ on the September 2009 futures. The net price paid is therefore

$$388.00 + 0.8 \times 8.00 - 0.8 \times 24.00 = 375.20$$

The hedging scheme succeeds in keeping the price paid in the range 371.40 to 375.20.

In October 2007 the initial margin requirement on the 128 contracts is $128 \times \$2,000$ or \$256,000. There is a margin call when the futures price drops by more than 2 cents. This happens to the March 2008 contract between October 2007 and February 2008, to the September 2008 contract between October 2007 and February 2008, and to the September 2008 contract between February 2008 and August 2008.

CHAPTER 4

Interest Rates

Notes for the Instructor

This chapter together with Chapters 6 and 7 emphasizes that, for a derivatives trader, risk-free rates are the rates derived from LIBOR markets, Eurodollar futures, and swap markets. The reasons why derivatives traders do not use Treasury rates as risk-free rates are outlined in Business Snapshot 4.1. Chapter 7 continues this discussion by explaining that swap rates have very little credit risk because a bank can earn the swap rate by making a series of short term loans to AA-rated companies.

A new feature of this chapter is the expanded treatment of liquidity preference theory in Section 4.10.

I like to spend a some time explaining compounding frequency issues. I make it clear to students that we are talking about nothing more than a unit of measurement for interest rates. Moving from quarterly compounding to continuous compounding is like changing the unit of measurement of distance from miles to kilometers. When students are introduced to continuous compounding early in a course, I find they have very little difficulty with it.

The first part of the chapter discusses zero rates, bond valuation, bond yields, par yields, and the calculation of the Treasury zero curve. The slides mirror the examples in the text. When covering the bootstrap method to calculate the Treasury zero curve, I mention that Chapter 7 explains how the same procedure can be used to calculate the LIBOR/swap zero curve. I also point out that the bootstrap method is a very popular approach, but it is not the only one that is used in practice. For example, some analysts use cubic or exponential splines.

I spend some time on the relationship between spot and forward interest rates and combine this with a discussion of FRAs and theories of the term structure. I explain that it is possible to enter into transactions that lock in the forward rate for a future time period and then discuss the Orange County story (Business Snapshot 4.2). Orange County entered into contracts (often highly levered) that paid off if the forward rate was higher than the realized future spot rate (An example of such a contract is an FRA where fixed is received and floating is paid). This worked well in 1992 and 1993, but led to a huge loss in 1994.

Sections 4.8 and 4.9 cover duration and convexity. Duration is a widely used concept in derivatives markets. The chapter explains that the $\Delta B/B = -D\Delta y$ relationship holds when rates are continuously compounded. When some other compounding frequency is used the same relationship is true provided D is defined as the modified duration. I like to illustrate the truth of the duration relationship with numerical example similar to those in the text.

Problems 4.24 to 4.28 can be used as assignment questions. My favorites are 4.27 and 4.28.

QUESTIONS AND PROBLEMS

Problem 4.1.

A bank quotes you an interest rate of 14% per annum with quarterly compounding. What is the equivalent rate with (a) continuous compounding and (b) annual compounding?

(a) The rate with continuous compounding is

$$4 \ln \left(1 + \frac{0.14}{4}\right) = 0.1376$$

or 13.76% per annum.

(b) The rate with annual compounding is

$$\left(1 + \frac{0.14}{4}\right)^4 - 1 = 0.1475$$

or 14.75% per annum.

Problem 4.2.

What is meant by LIBOR and LIBID. Which is higher?

LIBOR is the London InterBank Offered Rate. It is the rate a bank quotes for deposits it is prepared to place with other banks. LIBID is the London InterBank Bid rate. It is the rate a bank quotes for deposits from other banks. LIBOR is greater than LIBID.

Problem 4.3.

The six-month and one-year zero rates are both 10% per annum. For a bond that has a life of 18 months and pays a coupon of 8% per annum (with semiannual payments and one having just been made), the yield is 10.4% per annum. What is the bond's price? What is the 18-month zero rate? All rates are quoted with semiannual compounding.

Suppose the bond has a face value of \$100. Its price is obtained by discounting the cash flows at 10.4%. The price is

$$\frac{4}{1.052} + \frac{4}{1.052^2} + \frac{104}{1.052^3} = 96.74$$

If the 18-month zero rate is R, we must have

$$\frac{4}{1.05} + \frac{4}{1.05^2} + \frac{104}{(1 + R/2)^3} = 96.74$$

which gives $R = 10.42\%$.

Problem 4.4.

An investor receives \$1,100 in one year in return for an investment of \$1,000 now. Calculate the percentage return per annum with a) Annual compounding, b) Semiannual compounding, c) Monthly compounding and d) Continuous compounding.

(a) With annual compounding the return is

$$\frac{1100}{1000} - 1 = 0.1$$

or 10% per annum.

(b) With semi-annual compounding the return is R where

$$1000\left(1 + \frac{R}{2}\right)^2 = 1100$$

i.e.,

$$1 + \frac{R}{2} = \sqrt{1.1} = 1.0488$$

so that $R = 0.0976$. The percentage return is therefore 9.76% per annum.

(c) With monthly compounding the return is R where

$$1000\left(1 + \frac{R}{12}\right)^{12} = 1100$$

i.e.,

$$\left(1 + \frac{R}{12}\right) = \sqrt[12]{1.1} = 1.00797$$

so that $R = 0.0957$ The percentage return is therefore 9.57% per annum.

(d) With continuous compounding the return is R where:

$$1000e^R = 1100$$

i.e.,

$$e^R = 1.1$$

so that $R = \ln 1.1 = 0.0953$. The percentage return is therefore 9.53% per annum.

Problem 4.5.

Suppose that zero interest rates with continuous compounding are as follows:

Maturity (months)	*Rate (% per annum)*
3	*8.0*
6	*8.2*
9	*8.4*
12	*8.5*
15	*8.6*
18	*8.7*

Calculate forward interest rates for the second, third, fourth, fifth, and sixth quarters.

The forward rates with continuous compounding are as follows

Qtr 2:	8.4%
Qtr 3:	8.8%
Qtr 4:	8.8%
Qtr 5:	9.0%
Qtr 6:	9.2%

Problem 4.6.

Assuming that zero rates are as in Problem 4.5, what is the value of an FRA that enables the holder to earn 9.5% for a three-month period starting in one year on a principal of $1,000,000? The interest rate is expressed with quarterly compounding.

The forward rate is 9.0% with continuous compounding or 9.102% with quarterly compounding. From equation (4.9), the value of the FRA is therefore

$$[1,000,000 \times 0.25 \times (0.095 - 0.09102)]e^{-0.086 \times 1.25} = 893.56$$

or $893.56.

Problem 4.7.

The term structure of interest rates is upward sloping. Put the following in order of magnitude:

a. *The five-year zero rate*
b. *The yield on a five-year coupon-bearing bond*
c. *The forward rate corresponding to the period between 4.75 and 5 years in the future*

What is the answer to this question when the term structure of interest rates is downward sloping?

When the term structure is upward sloping, $c > a > b$. When it is downward sloping, $b > a > c$.

Problem 4.8.

What does duration tell you about the sensitivity of a bond portfolio to interest rates. What are the limitations of the duration measure?

Duration provides information about the effect of a small parallel shift in the yield curve on the value of a bond portfolio. The percentage decrease in the value of the portfolio equals the duration of the portfolio multiplied by the amount by which interest rates are increased in the small parallel shift. The duration measure has the following limitation. It applies only to parallel shifts in the yield curve that are small.

Problem 4.9.

What rate of interest with continuous compounding is equivalent to 15% per annum with monthly compounding?

The rate of interest is R where:

$$e^R = \left(1 + \frac{0.15}{12}\right)^{12}$$

i.e.,

$$R = 12 \ln \left(1 + \frac{0.15}{12}\right)$$
$$= 0.1491$$

The rate of interest is therefore 14.91% per annum.

Problem 4.10.

A deposit account pays 12% per annum with continuous compounding, but interest is actually paid quarterly. How much interest will be paid each quarter on a \$10,000 deposit?

The equivalent rate of interest with quarterly compounding is R where

$$e^{0.12} = \left(1 + \frac{R}{4}\right)^4$$

or

$$R = 4(e^{0.03} - 1) = 0.1218$$

The amount of interest paid each quarter is therefore:

$$10,000 \times \frac{0.1218}{4} = 304.55$$

or \$304.55.

Problem 4.11.

Suppose that 6-month, 12-month, 18-month, 24-month, and 30-month zero rates are 4%, 4.2%, 4.4%, 4.6%, and 4.8% per annum with continuous compounding respectively. Estimate the cash price of a bond with a face value of 100 that will mature in 30 months pays a coupon of 4% per annum semiannually.

The bond pays \$2 in 6, 12, 18, and 24 months, and \$102 in 30 months. The cash price is

$$2e^{-0.04\times0.5} + 2e^{-0.042\times1.0} + 2e^{-0.044\times1.5} + 2e^{-0.046\times2} + 102e^{-0.048\times2.5} = 98.04$$

Problem 4.12.

A three-year bond provides a coupon of 8% semiannually and has a cash price of 104. What is the bond's yield?

The bond pays \$4 in 6, 12, 18, 24, and 30 months, and \$104 in 36 months. The bond yield is the value of y that solves

$$4e^{-0.5y} + 4e^{-1.0y} + 4e^{-1.5y} + 4e^{-2.0y} + 4e^{-2.5y} + 104e^{-3.0y} = 104$$

Using the *Goal Seek* tool in Excel $y = 0.06407$ or 6.407%.

Problem 4.13.

Suppose that the 6-month, 12-month, 18-month, and 24-month zero rates are 5%, 6%, 6.5%, and 7% respectively. What is the two-year par yield?

Using the notation in the text, $m = 2$, $d = e^{-0.07\times2} = 0.8694$. Also

$$A = e^{-0.05\times0.5} + e^{-0.06\times1.0} + e^{-0.065\times1.5} + e^{-0.07\times2.0} = 3.6935$$

The formula in the text gives the par yield as

$$\frac{(100 - 100 \times 0.8694) \times 2}{3.6935} = 7.072$$

To verify that this is correct we calculate the value of a bond that pays a coupon of 7.072% per year (that is 3.5365 every six months). The value is

$$3.536e^{-0.05\times0.5} + 3.5365e^{-0.06\times1.0} + 3.536e^{-0.065\times1.5} + 103.536e^{-0.07\times2.0} = 100$$

verifying that 7.072% is the par yield.

Problem 4.14.

Suppose that zero interest rates with continuous compounding are as follows:

Maturity (years)	*Rate (% per annum)*
1	*2.0*
2	*3.0*
3	*3.7*
4	*4.2*
5	*4.5*

Calculate forward interest rates for the second, third, fourth, and fifth years.

The forward rates with continuous compounding are as follows:

Year 2:	4.0%
Year 3:	5.1%
Year 4:	5.7%
Year 5:	5.7%

Problem 4.15.

Use the rates in Problem 4.14 to value an FRA where you will pay 5% for the third year on $1 million.

The forward rate is 5.1% with continuous compounding or $e^{0.051\times 1} - 1 = 5.232\%$ with annual compounding. The 3-year interest rate is 3.7% with continuous compounding. From equation (4.10), the value of the FRA is therefore

$$[1,000,000 \times (0.05232 - 0.05) \times 1]e^{-0.037\times 3} = 2,078.85$$

or $2,078.85.

Problem 4.16.

A 10-year, 8% coupon bond currently sells for $90. A 10-year, 4% coupon bond currently sells for $80. What is the 10-year zero rate? (Hint: Consider taking a long position in two of the 4% coupon bonds and a short position in one of the 8% coupon bonds.)

Taking a long position in two of the 4% coupon bonds and a short position in one of the 8% coupon bonds leads to the following cash flows

$$\text{Year } 0: 90 - 2 \times 80 = -70$$
$$\text{Year } 10: 200 - 100 = 100$$

because the coupons cancel out. $100 in 10 years time is equivalent to $70 today. The 10-year rate, R, (continuously compounded) is therefore given by

$$100 = 70e^{10R}$$

The rate is

$$\frac{1}{10}\ln\frac{100}{70} = 0.0357$$

or 3.57% per annum.

Problem 4.17.

Explain carefully why liquidity preference theory is consistent with the observation that the term structure of interest rates tends to be upward sloping more often than it is downward sloping.

If long-term rates were simply a reflection of expected future short-term rates, we would expect the term structure to be downward sloping as often as it is upward sloping. (This is based on the assumption that half of the time investors expect rates to increase and half of the time investors expect rates to decrease). Liquidity preference theory argues that long term rates are high relative to expected future short-term rates. This means that the term structure should be upward sloping more often than it is downward sloping.

Problem 4.18.

"When the zero curve is upward sloping, the zero rate for a particular maturity is greater than the par yield for that maturity. When the zero curve is downward sloping the reverse is true." Explain why this is so.

The par yield is the yield on a coupon-bearing bond. The zero rate is the yield on a zero-coupon bond. When the yield curve is upward sloping, the yield on an N-year coupon-bearing bond is less than the yield on an N-year zero-coupon bond. This is because the coupons are discounted at a lower rate than the N-year rate and drag the yield down below this rate. Similarly, when the yield curve is downward sloping, the yield on an N-year coupon bearing bond is higher than the yield on an N-year zero-coupon bond.

Problem 4.19.

Why are U.S. Treasury rates significantly lower that other rates that are close to risk free?

There are three reasons (see Business Snapshot 4.1).

(i) Treasury bills and Treasury bonds must be purchased by financial institutions to fulfill a variety of regulatory requirements. This increases demand for these Treasury instruments driving the price up and the yield down.

(ii)The amount of capital a bank is required to hold to support an investment in Treasury bills and bonds is substantially smaller than the capital required to support a similar investment in other very-low-risk instruments.

(iii) In the United States, Treasury instruments are given a favorable tax treatment compared with most other fixed-income investments because they are not taxed at the state level.

Problem 4.20.

Why does a loan in the repo market involve very little credit risk?

A repo is a contract where an investment dealer who owns securities agrees to sell them to another company now and buy them back later at a slightly higher price. The other company is providing a loan to the investment dealer. This loan involves very little credit risk. If the borrower does not honor the agreement, the lending company simply keeps the securities. If the lending company does not keep to its side of the agreement, the original owner of the securities keeps the cash.

Problem 4.21.

Explain why an FRA is equivalent to the exchange of a floating rate of interest for a fixed rate of interest.

A FRA is an agreement that a certain specified interest rate, R_K, will apply to a certain principal, L, for a certain specified future time period. Suppose that the rate observed in the market for the future time period at the beginning of the time period proves to be R_M. If the FRA is an agreement that R_K will apply when the principal is invested, the holder of the FRA can borrow the principal at R_M and then invest it at R_K. The net cash flow at the end of the period is then an inflow of $R_K L$ and an outflow of $R_M L$. If the FRA is an agreement that R_K will apply when the principal is borrowed, the holder of the FRA can invest the borrowed principal at R_M. The net cash flow at the end of the period is then an inflow of $R_M L$ and an outflow of $R_K L$. In either case we see that the FRA involves the exchange of a fixed rate of interest, R_K, on the principal of L for the floating rate of interest observed in the market, R_M.

Problem 4.22.

A five-year bond with a yield of 11% (continuously compounded) pays an 8% coupon at the end of each year.

a. *What is the bond's price?*
b. *What is the bond's duration?*
c. *Use the duration to calculate the effect on the bond's price of a 0.2% decrease in its yield.*
d. *Recalculate the bond's price on the basis of a 10.8% per annum yield and verify that the result is in agreement with your answer to (c).*

(a) The bond's price is

$$8e^{-0.11} + 8e^{-0.11\times2} + 8e^{-0.11\times3} + 8e^{-0.11\times4} + 108e^{-0.11\times5} = 86.80$$

(b) The bond's duration is

$$\frac{1}{86.80}\left[8e^{-0.11} + 2\times 8e^{-0.11\times2} + 3\times 8e^{-0.11\times3} + 4\times 8e^{-0.11\times4} + 5\times 108e^{-0.11\times5}\right]$$

$$= 4.256 \text{ years}$$

(c) Since, with the notation in the chapter

$$\Delta B = -BD\Delta y$$

the effect on the bond's price of a 0.2% decrease in its yield is

$$86.80 \times 4.256 \times 0.002 = 0.74$$

The bond's price should increase from 86.80 to 87.54.

(d) With a 10.8% yield the bond's price is

$$8e^{-0.108} + 8e^{-0.108\times2} + 8e^{-0.108\times3} + 8e^{-0.108\times4} + 108e^{-0.108\times5} = 87.54$$

This is consistent with the answer in (c).

Problem 4.23.

The cash prices of six-month and one-year Treasury bills are 94.0 and 89.0. A 1.5-year bond that will pay coupons of $4 every six months currently sells for $94.84. A two-year bond that will pay coupons of $5 every six months currently sells for $97.12. Calculate the six-month, one-year, 1.5-year, and two-year zero rates.

The 6-month rate (with continuous compounding) is $2\ln(1+6/94) = 12.38\%$. The 12-month rate is $\ln(1+11/89) = 11.65\%$.

For the 1.5-year bond we must have

$$4e^{-0.1238\times0.5} + 4e^{-0.1165\times1.0} + 104e^{-1.5R} = 94.84$$

where R is the 1.5-year spot rate. It follows that

$$3.76 + 3.56 + 104e^{-1.5R} = 94.84$$

$$e^{-1.5R} = 0.8415$$

$$R = 0.115$$

or 11.5%. For the 2-year bond we must have

$$5e^{-0.1238\times0.5} + 5e^{-0.1165\times1.0} + 5e^{-0.115\times1.5} + 105e^{-2R} = 97.12$$

where R is the 2-year spot rate. It follows that

$$e^{-2R} = 0.7977$$
$$R = 0.113$$

or 11.3%.

ASSIGNMENT QUESTIONS

Problem 4.24.

An interest rate is quoted as 5% per annum with semiannual compounding. What is the equivalent rate with (a) annual compounding, (b) monthly compounding, and (c) continuous compounding.

(a) With annual compounding the rate is $1.025^2 - 1 = 0.050625$ or 5.0625%

(b) With monthly compounding the rate is $12 \times (1.025^{1/6} - 1) = 0.04949$ or 4.949%.

(c) With continuous compounding the rate is $2 \times \ln 1.025 = 0.04939$or 4.939%.

Problem 4.25.

The 6-month, 12-month. 18-month,and 24-month zero rates are 4%, 4.5%, 4.75%, and 5% with semiannual compounding.
(a) What are the rates with continuous compounding?
(b) What is the forward rate for the six-month period beginning in 18 months
(c) What is the value of an FRA that promises to pay you 6% (compounded semiannually) on a principal of $1 million for the six-month period starting in 18 months?

(a) With continuous compounding the 6-month rate is $2\ln 1.02 = 0.039605$ or 3.961%. The 12-month rate is $2\ln 1.0225 = 0.044501$ or 4.4501%. The 18-month rate is $2\ln 1.02375 = 0.046945$ or 4.6945%. The 24-month rate is $2\ln 1.025 = 0.049385$ or 4.9385%.

(b) The forward rate (expressed with continuous compounding) is from equation (4.5)

$$\frac{4.9385 \times 2 - 4.6945 \times 1.5}{0.5}$$

or 5.6707%. When expressed with semiannual compounding this is $2(e^{0.056707\times 0.5} - 1) = 0.057518$ or 5.7518%.

(c) The value of an FRA that where you will receive 6% for the six month period starting in 18 months is from equation (4.9)

$$1,000,000 \times (0.06 - 0.057518) \times 0.5e^{-0.049385\times 2} = 1,124$$

or $1,124.

Problem 4.26.

What is the two-year par yield when the zero rates are as in Problem 4.25? What is the yield on a two-year bond that pays a coupon equal to the par yield?

The value, A of an annuity paying off $1 every six months is

$$e^{-0.039605\times 0.5} + e^{-0.044501\times 1} + e^{-0.046945\times 1.5} + e^{-0.049385\times 2} = 3.7748$$

The present value of $1 received in two years, d, is $e^{-0.049385\times 2} = 0.90595$. From the formula in Section 4.4 the par yield is

$$\frac{(100 - 100 \times 0.90595) \times 2}{3.7748} = 4.983$$

or 4.983%.

Problem 4.27.

The following table gives the prices of bonds

Bond Principal ($)	*Time to Maturity (years)*	*Annual Coupon ($)**	*Bond Price ($)*
100	*0.50*	*0.0*	*98*
100	*1.00*	*0.0*	*95*
100	*1.50*	*6.2*	*101*
100	*2.00*	*8.0*	*104*

** Half the stated coupon is assumed to be paid every six months.*

a. *Calculate zero rates for maturities of 6 months, 12 months, 18 months, and 24 months.*
b. *What are the forward rates for the periods: 6 months to 12 months, 12 months to 18 months, 18 months to 24 months?*
c. *What are the 6-month, 12-month, 18-month, and 24-month par yields for bonds that provide semiannual coupon payments?*
d. *Estimate the price and yield of a two-year bond providing a semiannual coupon of 7% per annum.*

(a) The zero rate for a maturity of six months, expressed with continuous compounding is $2\ln(1+2/98) = 4.0405\%$. The zero rate for a maturity of one year, expressed with continuous compounding is $\ln(1+5/95) = 5.1293$. The 1.5-year rate is R where

$$3.1e^{-0.040405\times0.5} + 3.1e^{-0.051293\times1} + 103.1e^{-R\times1.5} = 101$$

The solution to this equation is $R = 0.054429$. The 2.0-year rate is R where

$$4e^{-0.040405\times0.5} + 4e^{-0.051293\times1} + 4e^{-0.054429\times1.5} + 104e^{-R\times2} = 104$$

The solution to this equation is $R = 0.058085$. These results are shown in the table below

Maturity (years)	Zero Rate (%)	Forward Rate (%)	Par Yield semi ann. (%)	Par Yield cont. comp.
0.5	4.0405	4.0405	4.0816	4.0405
1.0	5.1293	6.2181	5.1813	5.1154
1.5	5.4429	6.0700	5.4986	5.4244
2.0	5.8085	6.9054	5.8620	5.7778

(b) The continuously compounded forward rates calculated using equation (4.5) are shown in the third column of the table

(c) The par yield, expressed with semiannual compounding, can be calculated from the formula in Section 4.4. It is shown in the fourth column of the table. In the fifth column of the table it is converted to continuous compounding

(d) The price of the bond is

$$3.5e^{-0.040405\times0.5}+3.5e^{-0.051293\times1}+3.5e^{-0.054429\times1.5}+103.5e^{-0.058085\times2}=102.13$$

The yield on the bond, y satisfies

$$3.5e^{-y\times0.5}+3.5e^{-y\times1.0}+3.5e^{-y\times1.5}+103.5e^{-y\times2.0}=102.13$$

The solution to this equation is $y=0.057723$. The bond yield is therefore 5.7723%.

Problem 4.28.

Portfolio A consists of a one-year zero-coupon bond with a face value of $2,000 and a 10-year zero-coupon bond with a face value of $6,000. Portfolio B consists of a 5.95-year zero-coupon bond with a face value of $5,000. The current yield on all bonds is 10% per annum.

a. *Show that both portfolios have the same duration.*

b. *Show that the percentage changes in the values of the two portfolios for a 0.1% per annum increase in yields are the same.*

c. *What are the percentage changes in the values of the two portfolios for a 5% per annum increase in yields?*

(a) The duration of Portfolio A is

$$\frac{1\times2000e^{-0.1\times1}+10\times6000e^{-0.1\times10}}{2000e^{-0.1\times1}+6000e^{-0.1\times10}}=5.95$$

Since this is also the duration of Portfolio B, the two portfolios do have the same duration.

(b) The value of Portfolio A is

$$2000e^{-0.1}+6000e^{-0.1\times10}=4016.95$$

When yields increase by 10 basis points its value becomes

$$2000e^{-0.101}+6000e^{-0.101\times10}=3993.18$$

The percentage decrease in value is

$$\frac{23.77\times100}{4016.95}=0.59\%$$

The value of Portfolio B is

$$5000e^{-0.1\times5.95}=2757.81$$

When yields increase by 10 basis points its value becomes

$$5000e^{-0.101\times5.95}=2741.45$$

The percentage decrease in value is

$$\frac{16.36 \times 100}{2757.81} = 0.59\%$$

The percentage changes in the values of the two portfolios for a 10 basis point increase in yields are therefore the same.

(c) When yields increase by 5% the value of Portfolio A becomes

$$2000e^{-0.15} + 6000e^{-0.15 \times 10} = 3060.20$$

and the value of Portfolio B becomes

$$5000e^{-0.15 \times 5.95} = 2048.15$$

The percentage reduction in the values of the two portfolios are:

$$\text{Portfolio A:} \quad \frac{956.75}{4016.95} \times 100 = 23.82$$

$$\text{Portfolio B:} \quad \frac{709.66}{2757.81} \times 100 = 25.73$$

Since the percentage decline in value of Portfolio A is less than that of Portfolio B, Portfolio A has a greater convexity (see Figure 4.2 in text).

CHAPTER 5

Determination of Forward and Futures Prices

Notes for the Instructor

This chapter covers the relationship between forward/futures prices and spot prices. The approach used in the chapter is to produce results for forward prices first and then argue that futures prices are very close to forward prices. The early part of the chapter explains short selling and the difference between investment and consumption assets.

I usually go through the material in Section 5.4 fairly carefully to make sure that students understand the nature of the arguments that are used. (Business Snapshot 5.1, a description of Joseph Jett's trading at Kidder Peabody, helps to explain why equation 5.1 holds.) I then go through Sections 5.5 and 5.6 fairly quickly because the arguments in those sections are really just extensions of the argument in Section 5.4. However, it is necessary to explain carefully the difference between a known cash dividend and a known dividend yield.

When covering Section 5.7, I emphasize the distinction between f (the value of a long forward contract) and F_0 (the forward price). This often causes confusion. I like to go through Business Snapshot 5.2 to help students understand the issue.

If time permits I like to go through the material in the appendix. It reinforces the students' understanding of how futures contracts work and provides an interesting pure arbitrage argument.

The material in Sections 5.9 and 5.10 follows naturally from the material in Sections 5.4 to 5.6. I try to illustrate all of the formulas with numerical examples taken from current market quotes. The interpretation of a foreign currency as an investment providing a yield equal to the foreign risk-free rate needs to be explained carefully. I also like to spend some time discussing the fact that the variable underlying the CME Nikkei futures contract is not something that can be traded (see Business Snapshot 5.3).

It is important that students understand the distinction between assets that are held solely for investment by a significant number of investors and those that are not. This distinction is made right at the beginning of the chapter.

Section 5.14 ties the relationship between a futures prices and an expected future spot price to the notion of systematic risk, which will probably be familiar to students from other courses they have taken.

Problems 5.26 and 5.28 can be used for discussion in class. Problems 5.24, 5.25, and 5.27 can be used as assignment questions. My favorites are 5.25 and 5.27.

QUESTIONS AND PROBLEMS

Problem 5.1.

Explain what happens when an investor shorts a certain share.

The investor's broker borrows the shares from another client's account and sells them in the usual way. To close out the position, the investor must purchase the shares. The broker then replaces them in the account of the client from whom they were borrowed. The party with the short position must remit to the broker dividends and other income paid on the shares. The broker transfers these funds to the account of the client from whom the shares were borrowed. Occasionally the broker runs out of places from which to borrow the shares. The investor is then short squeezed and has to close out the position immediately.

Problem 5.2.

What is the difference between the forward price and the value of a forward contract?

The forward price of an asset today is the price at which you would agree to buy or sell the asset at a future time. The value of a forward contract is zero when you first enter into it. As time passes the underlying asset price changes and the value of the contract may become positive or negative.

Problem 5.3.

Suppose that you enter into a six-month forward contract on a non-dividend-paying stock when the stock price is \$30 and the risk-free interest rate (with continuous compounding) is 12% per annum. What is the forward price?

The forward price is

$$30e^{0.12\times 0.5} = \$31.86$$

Problem 5.4.

A stock index currently stands at 350. The risk-free interest rate is 8% per annum (with continuous compounding) and the dividend yield on the index is 4% per annum. What should the futures price for a four-month contract be?

The futures price is

$$350e^{(0.08-0.04)\times 0.3333} = \$354.7$$

Problem 5.5.

Explain carefully why the futures price of gold can be calculated from its spot price and other observable variables whereas the futures price of copper cannot.

Gold is an investment asset. If the futures price is too high, investors will find it profitable to increase their holdings of gold and short futures contracts. If the futures price is too low, they will find it profitable to decrease their holdings of gold and go long in the futures market. Copper is a consumption asset. If the futures price is too high, a strategy of buy copper and short futures works. However, because investors do not in general hold the asset, the strategy of sell copper and buy futures is not available to them. There is therefore an upper bound, but no lower bound, to the futures price.

Problem 5.6.

Explain carefully the meaning of the terms convenience yield and cost of carry. What is the relationship between futures price, spot price, convenience yield, and cost of carry?

Convenience yield measures the extent to which there are benefits obtained from ownership of the physical asset that are not obtained by owners of long futures contracts. The *cost of carry* is the interest cost plus storage cost less the income earned. The futures price, F_0, and spot price, S_0, are related by

$$F_0 = S_0 e^{(c-y)T}$$

where c is the cost of carry, y is the convenience yield, and T is the time to maturity of the futures contract.

Problem 5.7.

Explain why a foreign currency can be treated as an asset providing a known yield.

A foreign currency provides a known interest rate, but the interest is received in the foreign currency. The value in the domestic currency of the income provided by the foreign currency is therefore known as a percentage of the value of the foreign currency. This means that the income has the properties of a known yield.

Problem 5.8.

Is the futures price of a stock index greater than or less than the expected future value of the index? Explain your answer.

The futures price of a stock index is always less than the expected future value of the index. This follows from Section 5.14 and the fact that the index has positive systematic risk. For an alternative argument, let μ be the expected return required by investors on the index so that $E(S_T) = S_0 e^{(\mu-q)T}$. Because $\mu > r$ and $F_0 = S_0 e^{(r-q)T}$, it follows that $E(S_T) > F_0$.

Problem 5.9.

A one-year long forward contract on a non-dividend-paying stock is entered into when the stock price is \$40 and the risk-free rate of interest is 10% per annum with continuous compounding.

a. What are the forward price and the initial value of the forward contract?

b. Six months later, the price of the stock is \$45 and the risk-free interest rate is still 10%. What are the forward price and the value of the forward contract?

(a) The forward price, F_0, is given by equation (5.1) as:

$$F_0 = 40e^{0.1\times1} = 44.21$$

or \$44.21. The initial value of the forward contract is zero.

(b) The delivery price K in the contract is \$44.21. The value of the contract, f, after six months is given by equation (5.5) as:

$$f = 45 - 44.21e^{-0.1 \times 0.5}$$

$$= 2.95$$

i.e., it is \$2.95. The forward price is:

$$45e^{0.1 \times 0.5} = 47.31$$

or \$47.31.

Problem 5.10.

The risk-free rate of interest is 7% per annum with continuous compounding, and the dividend yield on a stock index is 3.2% per annum. The current value of the index is 150. What is the six-month futures price?

Using equation (5.3) the six month futures price is

$$150e^{(0.07-0.032)\times 0.5} = 152.88$$

or \$152.88.

Problem 5.11.

Assume that the risk-free interest rate is 9% per annum with continuous compounding and that the dividend yield on a stock index varies throughout the year. In February, May, August, and November, dividends are paid at a rate of 5% per annum. In other months, dividends are paid at a rate of 2% per annum. Suppose that the value of the index on July 31 is 1,300. What is the futures price for a contract deliverable on December 31 of the same year?

The futures contract lasts for five months. The dividend yield is 2% for three of the months and 5% for two of the months. The average dividend yield is therefore

$$\frac{1}{5}(3 \times 2 + 2 \times 5) = 3.2\%$$

The futures price is therefore

$$1300e^{(0.09-0.032)\times 0.4167} = 1,331.80$$

or \$1331.80.

Problem 5.12.

Suppose that the risk-free interest rate is 10% per annum with continuous compounding and that the dividend yield on a stock index is 4% per annum. The index is standing at 400, and the futures price for a contract deliverable in four months is 405. What arbitrage opportunities does this create?

The theoretical futures price is

$$400e^{(0.10-0.04)\times 4/12} = 408.08$$

The actual futures price is only 405. This shows that the index futures price is too low relative to the index. The correct arbitrage strategy is

1. Buy futures contracts
2. Short the shares underlying the index.

Problem 5.13.

Estimate the difference between short-term interest rates in Mexico and the United States on January 8, 2007 from the information in Table 5.4.

The settlement prices for the futures contracts are

Jan	0.91250
Mar	0.91025

The March 2007 price is about 0.25% below the January 2007 price. This suggests that the short-term interest rate in the Mexico exceeded short-term interest rates in the United States by about 0.25% per two months or about 1.5% per year.

Problem 5.14.

The two-month interest rates in Switzerland and the United States are 2% and 5% per annum, respectively, with continuous compounding. The spot price of the Swiss franc is \$0.8000. The futures price for a contract deliverable in two months is \$0.8100. What arbitrage opportunities does this create?

The theoretical futures price is

$$0.8000e^{(0.05-0.02)\times 2/12} = 0.8040$$

The actual futures price is too high. This suggests that an arbitrageur should buy Swiss francs and short Swiss francs futures.

Problem 5.15.

The spot price of silver is \$9 per ounce. The storage costs are \$0.24 per ounce per year payable quarterly in advance. Assuming that interest rates are 10% per annum for all maturities, calculate the futures price of silver for delivery in nine months.

The present value of the storage costs for nine months are

$$0.06 + 0.06e^{-0.10\times 0.25} + 0.06e^{-0.10\times 0.5} = 0.176$$

or \$0.176. The futures price is from equation (5.11) given by F_0 where

$$F_0 = (9.000 + 0.176)e^{0.1\times 0.75} = 9.89$$

i.e., it is \$9.89 per ounce.

Problem 5.16.

Suppose that F_1 and F_2 are two futures contracts on the same commodity with times to maturity, t_1 and t_2, where $t_2 > t_1$. Prove that

$$F_2 \leq F_1 e^{r(t_2-t_1)}$$

where r is the interest rate (assumed constant) and there are no storage costs. For the purposes of this problem, assume that a futures contract is the same as a forward contract.

If

$$F_2 > F_1 e^{r(t_2-t_1)}$$

an investor could make a riskless profit by

1. Taking a long position in a futures contract which matures at time t_1
2. Taking a short position in a futures contract which matures at time t_2

When the first futures contract matures, the asset is purchased for F_1 using funds borrowed at rate r. It is then held until time t_2 at which point it is exchanged for F_2 under the second contract. The costs of the funds borrowed and accumulated interest at time t_2 is $F_1 e^{r(t_2-t_1)}$ A positive profit of

$$F_2 - F_1 e^{r(t_2-t_1)}$$

is then realized at time t_2. This type of arbitrage opportunity cannot exist for long. Hence:

$$F_2 \leq F_1 e^{r(t_2-t_1)}$$

Problem 5.17.

When a known future cash outflow in a foreign currency is hedged by a company using a forward contract, there is no foreign exchange risk. When it is hedged using futures contracts, the marking-to-market process does leave the company exposed to some risk. Explain the nature of this risk. In particular, consider whether the company is better off using a futures contract or a forward contract when

a. The value of the foreign currency falls rapidly during the life of the contract
b. The value of the foreign currency rises rapidly during the life of the contract
c. The value of the foreign currency first rises and then falls back to its initial value
d. The value of the foreign currency first falls and
then rises back to its initial value

Assume that the forward price equals the futures price.

In total the gain or loss under a futures contract is equal to the gain or loss under the corresponding forward contract. However the timing of the cash flows is different. When the time value of money is taken into account a futures contract may prove to be more valuable or less valuable than a forward contract. Of course the company does not know in advance which will work out better. The long forward contract provides a perfect hedge. The long futures contract provides a slightly imperfect hedge.

(a) In this case the forward contract would lead to a slightly better outcome. The company will make a loss on its hedge. If the hedge is with a forward contract the whole of the loss will be realized at the end. If it is with a futures contract the loss will be realized day by day throughout the contract. On a present value basis the former is preferable.

(b) In this case the futures contract would lead to a slightly better outcome. The company will make a gain on the hedge. If the hedge is with a forward contract the gain will be realized at the end. If it is with a futures contract the gain will be realized day by day throughout the life of the contract. On a present value basis the latter is preferable.

(c) In this case the futures contract would lead to a slightly better outcome. This is because it would involve positive cash flows early and negative cash flows later.

(d) In this case the forward contract would lead to a slightly better outcome. This is because, in the case of the futures contract, the early cash flows would be negative and the later cash flow would be positive.

Problem 5.18.

It is sometimes argued that a forward exchange rate is an unbiased predictor of future exchange rates. Under what circumstances is this so?

From the discussion in Section 5.14 of the text, the forward exchange rate is an unbiased predictor of the future exchange rate when the exchange rate has no systematic risk. To have no systematic risk the exchange rate must be uncorrelated with the return on the market.

Problem 5.19.

Show that the growth rate in an index futures price equals the excess return of the index over the risk-free rate. Assume that the risk-free interest rate and the dividend yield are constant.

Suppose that F_0 is the futures price at time zero for a contract maturing at time T and F_1 is the futures price for the same contract at time t_1. It follows that

$$F_0 = S_0 e^{(r-q)T}$$

$$F_1 = S_1 e^{(r-q)(T-t_1)}$$

where S_0 and S_1 are the spot price at times zero and t_1, r is the risk-free rate, and q is the dividend yield. These equations imply that

$$\frac{F_1}{F_0} = \frac{S_1}{S_0} e^{-(r-q)t_1}$$

Define the excess return of the index over the risk-free rate as x. The total return is $r + x$ and the return realized in the form of capital gains is $r + x - q$. It follows that $S_1 = S_0 e^{(r+x-q)t_1}$ and the equation for F_1/F_0 reduces to

$$\frac{F_1}{F_0} = e^{xt_1}$$

which is the required result.

Problem 5.20.

Show that equation (5.3) is true by considering an investment in the asset combined with a short position in a futures contract. Assume that all income from the asset is reinvested in the asset. Use an argument similar to that in footnotes 2 and 4 and explain in detail what an arbitrageur would do if equation (5.3) did not hold.

Suppose we buy N units of the asset and invest the income from the asset in the asset. The income from the asset causes our holding in the asset to grow at a continuously compounded rate q. By time T our holding has grown to Ne^{qT} units of the asset. Analogously to footnotes 2 and 4 of Chapter 5, we therefore buy N units of the asset at time zero at a cost of S_0 per unit and enter into a forward contract to sell Ne^{qT} unit for F_0 per unit at time T. This generates the following cash flows:

Time 0: $-NS_0$

Time T: NF_0e^{qT}

Because there is no uncertainty about these cash flows, the present value of the time T inflow must equal the time zero outflow when we discount at the risk-free rate. This means that

$$NS_0 = (NF_0e^{qT})e^{-rT}$$

or

$$F_0 = S_0e^{(r-q)T}$$

This is equation (5.3).

If $F_0 > S_0e^{(r-q)T}$, an arbitrageur should borrow money at rate r and buy N units of the asset. At the same time the arbitrageur should enter into a forward contract to sell Ne^{qT} units of the asset at time T. As income is received, it is reinvested in the asset. At time T the loan is repaid and the arbitrageur makes a profit of $N(F_0e^{qT} - S_0e^{rT})$ at time T.

If $F_0 < S_0e^{(r-q)T}$, an arbitrageur should short N units of the asset investing the proceeds at rate r. At the same time the arbitrageur should enter into a forward contract to buy Ne^{qT} units of the asset at time T. When income is paid on the asset, the arbitrageur owes money on the short position. The investor meets this obligation from the cash proceeds of shorting further units. The result is that the number of units shorted grows at rate q to Ne^{qT}. The cumulative short position is closed out at time T and the arbitrageur makes a profit of $N(S_0e^{rT} - F_0e^{qT})$.

Problem 5.21.

Explain carefully what is meant by the expected price of a commodity on a particular future date. Suppose that the futures price of crude oil declines with the maturity of the contract at the rate of 2% per year. Assume that speculators tend to be short crude oil futures and hedgers tended to be long. What does the Keynes and Hicks argument imply about the expected future price of oil?

To understand the meaning of the expected future price of a commodity, suppose that there are N different possible prices at a particular future time: P_1, P_2, ..., P_N. Define

q_i as the (subjective) probability the price being P_i (with $q_1 + q_2 + \ldots + q_N = 1$). The expected future price is

$$\sum_{i=1}^{N} q_i P_i$$

Different people may have different expected future prices for the commodity. The expected future price in the market can be thought of as an average of the opinions of different market participants. Of course, in practice the actual price of the commodity at the future time may prove to be higher or lower than the expected price.

Keynes and Hicks argue that speculators on average make money from commodity futures trading and hedgers on average lose money from commodity futures trading. If speculators tend to have short positions in crude oil futures, the Keynes and Hicks argument implies that futures prices overstate expected future spot prices. If crude oil futures prices decline at 2% per year the Keynes and Hicks argument therefore implies an even faster decline for the expected price of crude oil.

Problem 5.22.

The Value Line Index is designed to reflect changes in the value of a portfolio of over 1,600 equally weighted stocks. Prior to March 9, 1988, the change in the index from one day to the next was calculated as the geometric average of the changes in the prices of the stocks underlying the index. In these circumstances, does equation (5.8) correctly relate the futures price of the index to its cash price? If not, does the equation overstate or understate the futures price?

When the geometric average of the price relatives is used, the changes in the value of the index do not correspond to changes in the value of a portfolio that is traded. Equation (5.8) is therefore no longer correct. The changes in the value of the portfolio is monitored by an index calculated from the arithmetic average of the prices of the stocks in the portfolio. Since the geometric average of a set of numbers is always less than the arithmetic average, equation (5.8) overstates the futures price. It is rumored that at one time (prior to 1988), equation (5.8) did hold for the Value Line Index. A major Wall Street firm was the first to recognize that this represented a trading opportunity. It made a financial killing by buying the stocks underlying the index and shorting the futures.

Problem 5.23.

A U.S. company is interested in using the futures contracts traded on the CME to hedge its Australian dollar exposure. Define r as the interest rate (all maturities) on the U.S. dollar and r_f as the interest rate (all maturities) on the Australian dollar. Assume that r and r_f are constant and that the company uses a contract expiring at time T to hedge an exposure at time t ($T > t$).

a. *Show that the optimal hedge ratio is*

$$e^{(r_f - r)(T-t)}$$

b. *Show that, when t is one day, the optimal hedge ratio is almost exactly S_0/F_0 where S_0 is the current spot price of the currency and F_0 is the current futures price of the currency for the contract maturing at time T.*

c. Show that the company can take account of the daily settlement of futures contracts for a hedge that lasts longer than one day by adjusting the hedge ratio so that it always equals the spot price of the currency divided by the futures price of the currency.

(a) The relationship between the futures price F_t and the spot price S_t at time t is

$$F_t = S_t e^{(r-r_f)(T-t)}$$

Suppose that the hedge ratio is h. The price obtained with hedging is

$$h(F_0 - F_t) + S_t$$

where F_0 is the initial futures price. This is

$$hF_0 + S_t - hS_t e^{(r-r_f)(T-t)}$$

If $h = e^{(r_f-r)(T-t)}$, this reduces to hF_0 and a zero variance hedge is obtained.

(b) When t is one day, h is approximately $e^{(r_f-r)T} = S_0/F_0$. The appropriate hedge ratio is therefore S_0/F_0.

(c) When a futures contract is used for hedging, the price movements in each day should in theory be hedged separately. This is because the daily settlement means that a futures contract is closed out and rewritten at the end of each day. From (b) the correct hedge ratio at any given time is, therefore, S/F where S is the spot price and F is the futures price. Suppose there is an exposure to N units of the foreign currency and M units of the foreign currency underlie one futures contract. With a hedge ratio of 1 we should trade N/M contracts. With a hedge ratio of S/F we should trade

$$\frac{SN}{FM}$$

contracts. In other words we should calculate the number of contracts that should be traded as the dollar value of our exposure divided by the dollar value of one futures contract (This is not the same as the dollar value of our exposure divided by the dollar value of the assets underlying one futures contract.) Since a futures contract is settled daily, we should in theory rebalance our hedge daily so that the outstanding number of futures contracts is always $(SN)/(FM)$. This is known as tailing the hedge. (See Section 3.4 of the text.)

ASSIGNMENT QUESTIONS

Problem 5.24.

A stock is expected to pay a dividend of \$1 per share in two months and in five months. The stock price is \$50, and the risk-free rate of interest is 8% per annum with *continuous compounding for all maturities. An investor has just taken a short position in a six-month forward contract on the stock.*

a. What are the forward price and the initial value of the forward contract?

b. Three months later, the price of the stock is \$48 and the risk-free rate of interest is still 8% per annum. What are the forward price and the value of the short position in the forward contract?

(a) The present value, I, of the income from the security is given by:

$$I = 1 \times e^{-0.08\times2/12} + 1 \times e^{-0.08\times5/12} = 1.9540$$

From equation (5.2) the forward price, F_0, is given by:

$$F_0 = (50 - 1.9540)e^{0.08\times0.5} = 50.01$$

or \$50.01. The initial value of the forward contract is (by design) zero. The fact that the forward price is very close to the spot price should come as no surprise. When the compounding frequency is ignored the dividend yield on the stock equals the risk-free rate of interest.

(b) In three months:

$$I = e^{-0.08\times2/12} = 0.9868$$

The delivery price, K, is 50.01. From equation (5.6) the value of the short forward contract, f, is given by

$$f = -(48 - 0.9868 - 50.01e^{-0.08\times3/12}) = 2.01$$

and the forward price is

$$(48 - 0.9868)e^{0.08\times3/12} = 47.96$$

Problem 5.25.

A bank offers a corporate client a choice between borrowing cash at 11% per annum and borrowing gold at 2% per annum. (If gold is borrowed, interest must be repaid in gold. Thus, 100 ounces borrowed today would require 102 ounces to be repaid in one year.) The risk-free interest rate is 9.25% per annum, and storage costs are 0.5% per annum. Discuss whether the rate of interest on the gold loan is too high or too low in relation to the rate of interest on the cash loan. The interest rates on the two loans are expressed with annual compounding. The risk-free interest rate and storage costs are expressed with continuous compounding.

My explanation of this problem to students usually goes as follows. Suppose that the price of gold is \$550 per ounce and the corporate client wants to borrow \$550,000. The client has a choice between borrowing \$550,000 in the usual way and borrowing 1,000 ounces of gold. If it borrows \$550,000 in the usual way, an amount equal to $550,000\times1.11 =$

$610,500 must be repaid. If it borrows 1,000 ounces of gold it must repay 1,020 ounces. In equation (5.12), $r = 0.0925$ and $u = 0.005$ so that the forward price is

$$550e^{(0.0925+0.005)\times 1} = 606.33$$

By buying 1,020 ounces of gold in the forward market the corporate client can ensure that the repayment of the gold loan costs

$$1,020 \times 606.33 = \$618,457$$

Clearly the cash loan is the better deal ($618,457 > 610,500$).

This argument shows that the rate of interest on the gold loan is too high. What is the correct rate of interest? Suppose that R is the rate of interest on the gold loan. The client must repay $1,000(1+R)$ ounces of gold. When forward contracts are used the cost of this is

$$1,000(1+R) \times 606.33$$

This equals the $610,500 required on the cash loan when $R = 0.688\%$. The rate of interest on the gold loan is too high by about 1.31%. However, this might be simply a reflection of the higher administrative costs incurred with a gold loan.

It is interesting to note that this is not an artificial question. Many banks are prepared to make gold loans at interest rates of about 2% per annum.

Problem 5.26.

A company that is uncertain about the exact date when it will pay or receive a foreign currency may try to negotiate with its bank a forward contract that specifies a period during which delivery can be made. The company wants to reserve the right to choose the exact delivery date to fit in with its own cash flows. Put yourself in the position of the bank. How would you price the product that the company wants?

It is likely that the bank will price the product on assumption that the company chooses the delivery date least favorable to the bank. If the foreign interest rate is higher than the domestic interest rate then

1. The earliest delivery date will be assumed when the company has a long position.
2. The latest delivery date will be assumed when the company has a short position.

If the foreign interest rate is lower than the domestic interest rate then

1. The latest delivery date will be assumed when the company has a long position.
2. The earliest delivery date will be assumed when the company has a short position.

If the company chooses a delivery which, from a purely financial viewpoint, is suboptimal the bank makes a gain.

Problem 5.27.

A trader owns gold as part of a long-term investment portfolio. The trader can buy gold for \$550 per ounce and sell gold for \$549 per ounce. The trader can borrow funds at 6% per year and invest funds at 5.5% per year. (Both interest rates are expressed with *annual compounding.) For what range of one-year forward prices of gold does the trader have no arbitrage opportunities? Assume there is no bid–offer spread for forward prices.*

Suppose that F_0 is the one-year forward price of gold. If F_0 is relatively high, the trader can borrow \$550 at 6%, buy one ounce of gold and enter into a forward contract to sell gold in one year for F_0. The profit made in one year is

$$F_0 - 550 \times 1.06 = F_0 - 583$$

If F_0 is relatively low, the trader can sell one ounce of gold for \$549, invest the proceeds at 5.5%, and enter into a forward contract to buy the gold back for F_0. The profit (relative to the position the trader would be in if the gold were held in the portfolio during the year) is

$$549 \times 1.055 - F_0 = 579.195 - F_0$$

This shows that there is no arbitrage opportunity if the forward price is between \$579.195 and \$583 per ounce.

Problem 5.28.

A company enters into a forward contract with a bank to sell a foreign currency for K_1 at time T_1. The exchange rate at time T_1 proves to be S_1 ($> K_1$). The company asks the bank if it can roll the contract forward until time T_2 ($> T_1$) rather than settle at time T_1. The bank agrees to a new delivery price, K_2. Explain how K_2 should be calculated.

The value of the contract to the bank at time T_1 is $S_1 - K_1$. The bank will choose K_2 so that the new (rolled forward) contract has a value of $S_1 - K_1$. This means that

$$S_1 e^{-r_f(T_2-T_1)} - K_2 e^{-r(T_2-T_1)} = S_1 - K_1$$

where r and r_f and the domestic and foreign risk-free rate observed at time T_1 and applicable to the period between time T_1 and T_2. This means that

$$K_2 = S_1 e^{(r-r_f)(T_2-T_1)} - (S_1 - K_1)e^{r(T_2-T_1)}$$

This equation shows that there are two components to K_2. The first is the forward price at time T_1. The second is an adjustment to the forward price equal to the bank's gain on the first part of the contract compounded forward at the domestic risk-free rate.

CHAPTER 6
Interest Rate Futures

Notes for the Instructor

This chapter discusses how interest rate futures contracts are quoted, how they work, and how they are used for hedging. I start by discussing the material in Sections 6.1 and 6.2 on day counts and how prices are quoted in the spot market. (It is fun to talk about Business Snapshot 6.1 when day count conventions are discussed.) I like to spend some time making sure students are comfortable with the Treasury bond futures contracts and the Eurodollar futures contract. In the case of the Treasury bond futures contract they should understand where conversion factors come from, the cheapest-to-deliver bond calculations, and the wild card play (see Business Snapshot 6.2). In the case of Eurodollar futures they should understand the quotation system, that the contract's value changes by $25 for each basis point change in the quote, and how the final cash settlement works. In the slides I have included a numerical example help explain this.

Students should also appreciate that a convexity adjustment is necessary to calculate a forward rate from a Eurodollar futures quote. They will not at this stage understand where equation 6.3 comes from, but they should understand that there are two reasons why forward and futures interest rates are different. The first is that futures are settled daily; forwards are not. The second is that futures (if not daily settled) would provide a payoff at the beginning of the period covered by the rate; forwards provide a payoff at the end of the period covered by the rate.

The final part of the chapter covers the use of interest rate futures for duration-based hedging. I usually illustrate this material with a numerical example.

I sometimes use Problem 6.23 in class to help explain how Eurodollar futures contracts work and the impact of day count conventions. (Without adjusting for the day count convention, the arbitrage opportunity appears to be the other way round.) Problems 6.24 and 6.26 can be used as assignment questions.

QUESTIONS AND PROBLEMS

Problem 6.1.

A U.S. Treasury bond pays a 7% coupon on January 7 and July 7. How much interest accrue per $100 of principal to the bond holder between July 7, 2009 and August 9, 2009? How would your answer be different if it were a corporate bond?

There are 33 calendar days between July 7, 2009 and August 9, 2009. There are 184 calendar days between July 7, 2009 and January 7, 2010. The interest earned per $100 of principal is therefore $3.5 \times 33/184 = \$0.6277$. For a corporate bond we assume 32 days between July 7 and August 9, 2009 and 180 days between July 7, 2009 and January 7, 2010. The interest earned is $3.5 \times 32/180 = \$0.6222$.

Problem 6.2.

It is January 9, 2009. The price of a Treasury bond with a 12% coupon that matures on October 12, 2020, is quoted as 102-07. What is the cash price?

There are 89 days between October 12, 2009, and January 9, 2010. There are 182 days between October 12, 2009, and April 12, 2010. The cash price of the bond is obtained by adding the accrued interest to the quoted price. The quoted price is $102\frac{7}{32}$ or 102.21875. The cash price is therefore

$$102.21875 + \frac{89}{182} \times 6 = \$105.15$$

Problem 6.3.

How is the conversion factor of a bond calculated by the Chicago Board of Trade? How is it used?

The conversion factor for a bond is equal to the quoted price the bond would have per dollar of principal on the first day of the delivery month on the assumption that the interest rate for all maturities equals 6% per annum (with semiannual compounding). The bond maturity and the times to the coupon payment dates are rounded down to the nearest three months for the purposes of the calculation. The conversion factor defines how much an investor with a short bond futures contract receives when bonds are delivered. If the conversion factor is 1.2345 the amount investor receives is calculated by multiplying 1.2345 by the most recent futures price and adding accrued interest.

Problem 6.4.

A Eurodollar futures price changes from 96.76 to 96.82. What is the gain or loss to an investor who is long two contracts?

The Eurodollar futures price has increased by 6 basis points. The investor makes a gain per contract of $25 \times 6 = \$150$ or \$300 in total.

Problem 6.5.

What is the purpose of the convexity adjustment made to Eurodollar futures rates? Why is the convexity adjustment necessary?

Suppose that a Eurodollar futures quote is 95.00. This gives a futures rate of 5% for the three-month period covered by the contract. The convexity adjustment is the amount by which futures rate has to be reduced to give an estimate of the forward rate for the period The convexity adjustment is necessary because a) the futures contract is settled daily and b) the futures contract expires at the beginning of the three months. Both of these lead to the futures rate being greater than the forward rate.

Problem 6.6.

The 350-day LIBOR rate is 3% with continuous compounding and the forward rate calculated from a Eurodollar futures contract that matures in 350 days is 3.2% with continuous compounding. Estimate the 440-day zero rate.

From equation (6.4) the rate is

$$\frac{3.2 \times 90 + 3 \times 350}{440} = 3.0409$$

or 3.0409%.

Problem 6.7.

It is January 30. You are managing a bond portfolio worth $6 million. The duration of the portfolio in six months will be 8.2 years. The September Treasury bond futures price is currently 108-15, and the cheapest-to-deliver bond will have a duration of 7.6 years in September. How should you hedge against changes in interest rates over the next six months?

The value of a contract is $108\frac{15}{32} \times 1,000 = \$108,468.75$. The number of contracts that should be shorted is

$$\frac{6,000,000}{108,468.75} \times \frac{8.2}{7.6} = 59.7$$

Rounding to the nearest whole number, 60 contracts should be shorted. The position should be closed out at the end of July.

Problem 6.8.

The price of a 90-day Treasury bill is quoted as 10.00. What continuously compounded return (on an actual/365 basis) does an investor earn on the Treasury bill for the 90-day period?

The cash price of the Treasury bill is

$$100 - \frac{90}{360} \times 10 = \$97.50$$

The annualized continuously compounded return is

$$\frac{365}{90} \ln\left(1 + \frac{2.5}{97.5}\right) = 10.27\%$$

Problem 6.9.

It is May 5, 2008. The quoted price of a government bond with a 12% coupon that matures on July 27, 2011, is 110-17. What is the cash price?

The number of days between January 27, 2008 and May 5, 2008 is 99. The number of days between January 27, 2008 and July 27, 2008 is 182. The accrued interest is therefore

$$6 \times \frac{99}{182} = 3.2637$$

The quoted price is 110.5312. The cash price is therefore

$$110.5312 + 3.2637 = 113.7949$$

or $113.79.

Problem 6.10.

Suppose that the Treasury bond futures price is 101-12. Which of the following four bonds is cheapest to deliver?

Bond	*Price*	*Conversion Factor*
1	*125-05*	*1.2131*
2	*142-15*	*1.3792*
3	*115-31*	*1.1149*
4	*144-02*	*1.4026*

The cheapest-to-deliver bond is the one for which

$$\text{Quoted Price} - \text{Futures Price} \times \text{Conversion Factor}$$

is least. Calculating this factor for each of the 4 bonds we get

$$\text{Bond 1: } 125.15625 - 101.375 \times 1.2131 = 2.178$$
$$\text{Bond 2: } 142.46875 - 101.375 \times 1.3792 = 2.652$$
$$\text{Bond 3: } 115.96875 - 101.375 \times 1.1149 = 2.946$$
$$\text{Bond 4: } 144.06250 - 101.375 \times 1.4026 = 1.874$$

Bond 4 is therefore the cheapest to deliver.

Problem 6.11.

It is July 30, 2009. The cheapest-to-deliver bond in a September 2009 Treasury bond futures contract is a 13% coupon bond, and delivery is expected to be made on September 30, 2009. Coupon payments on the bond are made on February 4 and August 4 each year. The term structure is flat, and the rate of interest with semiannual compounding is 12% per annum. The conversion factor for the bond is 1.5. The current quoted bond price is $110. Calculate the quoted futures price for the contract.

There are 176 days between February 4 and July 30 and 181 days between February 4 and August 4. The cash price of the bond is, therefore:

$$110 + \frac{176}{181} \times 6.5 = 116.32$$

The rate of interest with continuous compounding is $2\ln 1.06 = 0.1165$ or 11.65% per annum. A coupon of 6.5 will be received in 5 days (= 0.01370 years) time. The present value of the coupon is

$$6.5e^{-0.01370\times 0.1165} = 6.490$$

The futures contract lasts for 62 days (= 0.1699 years). The cash futures price if the contract were written on the 13% bond would be

$$(116.32 - 6.490)e^{0.1699\times 0.1165} = 112.03$$

At delivery there are 57 days of accrued interest. The quoted futures price if the contract were written on the 13% bond would therefore be

$$112.03 - 6.5 \times \frac{57}{184} = 110.01$$

Taking the conversion factor into account the quoted futures price should be:

$$\frac{110.01}{1.5} = 73.34$$

Problem 6.12.

An investor is looking for arbitrage opportunities in the Treasury bond futures market. What complications are created by the fact that the party with a short position can choose to deliver any bond with a maturity of over 15 years?

If the bond to be delivered and the time of delivery were known, arbitrage would be straightforward. When the futures price is too high, the arbitrageur buys bonds and shorts an equivalent number of bond futures contracts. When the futures price is too low, the arbitrageur sells bonds and goes long an equivalent number of bond futures contracts.

Uncertainty as to which bond will be delivered introduces complications. The bond that appears cheapest-to-deliver now may not in fact be cheapest-to-deliver at maturity. In the case where the futures price is too high, this is not a major problem since the party with the short position (i.e., the arbitrageur) determines which bond is to be delivered. In the case where the futures price is too low, the arbitrageur's position is far more difficult since he or she does not know which bond to buy; it is unlikely that a profit can be locked in for all possible outcomes.

Problem 6.13.

Suppose that the nine-month LIBOR interest rate is 8% per annum and the six-month LIBOR interest rate is 7.5% per annum (both with actual/365 and continuous compounding). Estimate the three-month Eurodollar futures price quote for a contract maturing in six months.

The forward interest rate for the time period between months 6 and 9 is 9% per annum with continuous compounding. This is because 9% per annum for three months when combined with $7\frac{1}{2}\%$ per annum for six months gives an average interest rate of 8% per annum for the nine-month period.

With quarterly compounding the forward interest rate is

$$4(e^{0.09/4} - 1) = 0.09102$$

or 9.102%. This assumes that the day count is actual/actual. With a day count of actual/360 the rate is $9.102 \times 360/365 = 8.977$. The three-month Eurodollar quote for a contract maturing in six months is therefore

$$100 - 8.977 = 91.02$$

This assumes no difference between futures and forward prices.

Problem 6.14.

Suppose that the 300-day LIBOR zero rate is 4% and Eurodollar quotes for contracts maturing in 300, 398 and 489 days are 95.83, 95.62, and 95.48. Calculate 398-day and 489-day LIBOR zero rates. Assume no difference between forward and futures rates for the purposes of your calculations.

The forward rates calculated form the first two Eurodollar futures are 4.17% and 4.38%. These are expressed with an actual/360 day count and quarterly compounding. With continuous compounding and an actual/365 day count they are $(365/90)\ln(1+0.0417/4) = 4.2060\%$ and $(365/90)\ln(1+0.0438/4) = 4.4167\%$. It follows from equation (6.4) that the 398 day rate is

$$\frac{4 \times 300 + 4.2060 \times 98}{398} = 4.0507$$

or 4.0507%. The 489 day rate is

$$\frac{4.0507 \times 398 + 4.4167 \times 91}{489} = 4.1188$$

or 4.1188%. We are assuming that the first futures rate applies to 98 days rather than the usual 91 days. The third futures quote is not needed.

Problem 6.15.

Suppose that a bond portfolio with a duration of 12 years is hedged using a futures contract in which the underlying asset has a duration of four years. What is likely to be the impact on the hedge of the fact that the 12-year rate is less volatile than the four-year rate?

Duration-based hedging schemes assume parallel shifts in the yield curve. Since the 12-year rate tends to move by less than the 4-year rate, the portfolio manager may find that he or she is over-hedged.

Problem 6.16.

Suppose that it is February 20 and a treasurer realizes that on July 17 the company will have to issue $5 million of commercial paper with a maturity of 180 days. If the paper were issued today, the company would realize $4,820,000. (In other words, the company would receive $4,820,000 for its paper and have to redeem it at $5,000,000 in 180 days' time.) The September Eurodollar futures price is quoted as 92.00. How should the treasurer hedge the company's exposure?

The company treasurer can hedge the company's exposure by shorting Eurodollar futures contracts. The Eurodollar futures position leads to a profit if rates rise and a loss if they fall.

The duration of the commercial paper is twice that of the Eurodollar deposit underlying the Eurodollar futures contract. The contract price of a Eurodollar futures contract is 980,000. The number of contracts that should be shorted is, therefore,

$$\frac{4,820,000}{980,000} \times 2 = 9.84$$

Rounding to the nearest whole number 10 contracts should be shorted.

Problem 6.17.

On August 1 a portfolio manager has a bond portfolio worth $10 million. The duration of the portfolio in October will be 7.1 years. The December Treasury bond futures price is currently 91-12 and the cheapest-to-deliver bond will have a duration of 8.8 years at maturity. How should the portfolio manager immunize the portfolio against changes in interest rates over the next two months?

The treasurer should short Treasury bond futures contract. If bond prices go down, this futures position will provide offsetting gains. The number of contracts that should be shorted is

$$\frac{10,000,000 \times 7.1}{91,375 \times 8.8} = 88.30$$

Rounding to the nearest whole number 88 contracts should be shorted.

Problem 6.18.

How can the portfolio manager change the duration of the portfolio to 3.0 years in Problem 6.17?

The answer in Problem 6.17 is designed to reduce the duration to zero. To reduce the duration from 7.1 to 3.0 instead of from 7.1 to 0, the treasurer should short

$$\frac{4.1}{7.1} \times 88.30 = 50.99$$

or 51 contracts.

Problem 6.19.

Between October 30, 2009, and November 1, 2009, you have a choice between owning a U.S. government bond paying a 12% coupon and a U.S. corporate bond paying a 12% coupon. Consider carefully the day count conventions discussed in this chapter and decide which of the two bonds you would prefer to own. Ignore the risk of default.

You would prefer to own the Treasury bond. Under the 30/360 day count convention there is one day between October 30, 2009 and November 1, 2009. Under the actual/actual (in period) day count convention, there are two days. Therefore you would earn approximately twice as much interest by holding the Treasury bond. This assumes that the quoted prices of the two bonds are the same.

Problem 6.20.

Suppose that a Eurodollar futures quote is 88 for a contract maturing in 60 days. What is the LIBOR forward rate for the 60- to 150-day period? Ignore the difference between futures and forwards for the purposes of this question.

The Eurodollar futures contract price of 88 means that the Eurodollar futures rate is 12% per annum. This is the forward rate for the 60- to 150-day period with quarterly compounding and an actual/360 day count convention.

Problem 6.21.

The three-month Eurodollar futures price for a contract maturing in six years is quoted as 95.20. The standard deviation of the change in the short-term interest rate in one year is 1.1%. Estimate the forward LIBOR interest rate for the period between 6.00 and 6.25 years in the future.

Using the notation of Section 6.4, $\sigma = 0.011$, $T_1 = 6$, and $T_2 = 6.25$. The convexity adjustment is

$$\frac{1}{2} \times 0.011^2 \times 6 \times 6.25 = 0.002269$$

or about 23 basis points. The futures rate is 4.8% with quarterly compounding and an actual/360 day count. $(365/90)ln(1.012) = 0.0484$ or 4.84% with continuous compounding and actual/365 day count. The forward rate is therefore $4.84 - 0.23 = 4.61\%$ with continuous compounding.

Problem 6.22.

Explain why the forward interest rate is less than the corresponding futures interest rate calculated from a Eurodollar futures contract.

Suppose that the contracts apply to the interest rate between times T_1 and T_2. There are two reasons for a difference between the forward rate and the futures rate. The first is that the futures contract is settled daily whereas the forward contract is settled once at time T_2. The second is that without daily settlement a futures contract would be settled at time T_1 not T_2. Both reasons tend to make the futures rate greater than the forward rate.

ASSIGNMENT QUESTIONS

Problem 6.23.

Assume that a bank can borrow or lend money at the same interest rate in the LIBOR market. The 90-day rate is 10% per annum, and the 180-day rate is 10.2% per annum, both expressed with continuous compounding and actual/actual day count. The Eurodollar futures price for a contract maturing in 91 days is quoted as 89.5. What arbitrage opportunities are open to the bank?

The Eurodollar futures contract price of 89.5 means that the Eurodollar futures rate is 10.5% per annum with quarterly compounding and an actual/360 day count. This becomes $10.5 \times 365/360 = 10.646\%$ with an actual/actual day count. This is

$$4 \ln(1 + 0.25 \times 0.10646) = 0.1051$$

or 10.51% with continuous compounding. The forward rate given by the 91-day rate and the 182-day rate is 10.4% with continuous compounding. This suggests the following arbitrage opportunity:

1. Buy Eurodollar futures.
2. Borrow 182-day money.
3. Invest the borrowed money for 91 days.

Problem 6.24.

A Canadian company wishes to create a Canadian LIBOR futures contract from a U.S. Eurodollar futures contract and forward contracts on foreign exchange. Using an example, explain how the company should proceed. For the purposes of this problem, assume that a futures contract is the same as a forward contract.

The U.S. Eurodollar futures contract maturing at time T enables an investor to lock in the forward rate for the period between T and T^* where T^* is three months later than T. If $\hat{r}$ is the forward rate, the U.S. dollar cash flows that can be locked in are

$$\begin{array}{lll} -Ae^{-\hat{r}(T^*-T)} & \text{at time} & T \\ +A & \text{at time} & T^* \end{array}$$

where A is the principal amount. To convert these to Canadian dollar cash flows, the Canadian company must enter into a short forward foreign exchange contract to sell Canadian dollars at time T and a long forward foreign exchange contract to buy Canadian dollars at time T^*. Suppose F and F^* are the forward exchange rates for contracts maturing at times T and T^*. (These represent the number of Canadian dollars per U.S. dollar.) The Canadian dollars to be sold at time T are

$$Ae^{-\hat{r}(T^*-T)}F$$

and the Canadian dollars to be purchased at time T^* are

$$AF^*$$

The forward contracts convert the U.S. dollar cash flows to the following Canadian dollar cash flows:

$$\begin{array}{lll} -Ae^{-\hat{r}(T^*-T)}F & \text{at time} & T \\ +AF^* & \text{at time} & T^* \end{array}$$

This is a Canadian dollar LIBOR futures contract where the principal amount is AF^*.

Problem 6.25.

The futures price for the June 2009 CBOT bond futures contract is 118-23.

a. *Calculate the conversion factor for a bond maturing on January 1, 2025, paying a coupon of 10%.*
b. *Calculate the conversion factor for a bond maturing on October 1, 2030, paying a coupon of 7%.*
c. *Suppose that the quoted prices of the bonds in (a) and (b) are 169.00 and 136.00, respectively. Which bond is cheaper to deliver?*
d. *Assuming that the cheapest-to-deliver bond is actually delivered, what is the cash price received for the bond?*

(a) On the first day of the delivery month the bond has 15 years and 7 months to maturity. The value of the bond assuming it lasts 15.5 years and all rates are 6% per annum with semiannual compounding is

$$\sum_{i=1}^{31} \frac{5}{1.03^i} + \frac{100}{1.03^{31}} = 140.00$$

The conversion factor is therefore 1.4000.

(b) On the first day of the delivery month the bond has 21 years and 4 months to maturity. The value of the bond assuming it lasts 21.25 years and all rates are 6% per annum with semiannual compounding is

$$\frac{1}{\sqrt{1.03}}\left[3.5 + \sum_{i=1}^{42} \frac{3.5}{1.03^i} + \frac{100}{1.03^{42}}\right] = 113.66$$

Subtracting the accrued interest of 1.75, this becomes 111.91. The conversion factor is therefore 1.1191.

(c) For the first bond, the quoted futures price times the conversion factor is

$$118.71825 \times 1.4000 = 166.2056$$

This is 2.7944 less than the quoted bond price. For the second bond, the quoted futures price times the conversion factor is

$$118.71825 \times 1.1191 = 132.8576$$

This is 3.1424 less than the quoted bond price. The first bond is therefore the cheapest to deliver.

(d) The price received for the bond is 166.2056 plus accrued interest.[1] There are 176 days between January 1, 2009 and June 25, 2009. There are 181 days between January 1, 2009 and July 1, 2009. The accrued interest is therefore

$$5 \times \frac{176}{181} = 4.8619$$

The cash price received for the bond is therefore 171.0675.

Problem 6.26.

A portfolio manager plans to use a Treasury bond futures contract to hedge a bond portfolio over the next three months. The portfolio is worth $100 million and will have a duration of 4.0 years in three months. The futures price is 122, and each futures contract is on $100,000 of bonds. The bond that is expected to be cheapest to deliver will have a duration of 9.0 years at the maturity of the futures contract. What position in futures contracts is required?

a. What adjustments to the hedge are necessary if after one month the bond that is expected to be cheapest to deliver changes to one with a duration of seven years?

b. Suppose that all rates increase over the next three months, but long-term rates increase less than short-term and medium-term rates. What is the effect of this on the performance of the hedge?

The number of short futures contracts required is

$$\frac{100,000,000 \times 4.0}{122,000 \times 9.0} = 364.3$$

Rounding to the nearest whole number 364 contracts should be shorted.

(a) This increases the number of contracts that should be shorted to

$$\frac{100,000,000 \times 4.0}{122,000 \times 7.0} = 468.4$$

or 468 when we round to the nearest whole number.

(b) In this case the gain on the short futures position is likely to be less than the loss on the bond portfolio. This is because the gain on the short futures position depends on the size of the movement in long-term rates and the loss on the bond portfolio depends on the size of the movement in medium-term rates. Duration-based hedging assumes that the movements in the two rates are the same.

[1] Note that the delivery date was not specified in the first printing of the book. We assume it is June 25.

CHAPTER 7
Swaps

Notes for the Instructor

This chapter covers the nature of swaps and how they are valued. I believe that it makes sense to teach swaps soon after forward contracts are covered because a swap is nothing more than a convenient way of bundling forward contracts. The growth of the swaps since the early 1980s makes them one of the most important derivative instruments. This chapter covers interest rate and currency swaps and provides a brief review of nonstandard swaps. More details on nonstandard swaps are in Chapter 32.

After explaining how swaps work and the way they can be used to transform assets and liabilities, I present the traditional comparative advantage argument for plain vanilla interest rate swaps and then proceed to explain why it is flawed. This usually generates a lively discussion. The key point is that the comparative advantage argument compares apples with oranges. Suppose a BBB-rated company wants to borrow at a fixed rate for five years and can choose between a fixed rate of 8% and a floating rate of LIBOR+1%. Borrowing floating and swapping to fixed appears attractive. But this ignores a key point. A fixed-rate loan will lead to exactly the same rate of interest applying each year for five years. By contrast, the spread over LIBOR on the floating-rate loan is usually guaranteed for only 6 months. If the creditworthiness of the company declines, the rate is liable to increase when the loan is rolled over. This means that borrowing floating and swapping to fixed subjects the BBB to "rollover risk". If a financial institution offers LIBOR+1% and guarantees that the spread over LIBOR will not change, we are comparing apples with apples. However, when a table similar to 7.4 is constructed, there is then found to be no comparative advantage.

A useful exercise is to take a situation such as that shown in Figure 7.7 and ask students to identify the credit risk and rollover risk of AAACorp, BBBCorp, and the financial institution.

This is the time when the nature of the LIBOR/swap zero curve can be explained to students. I go over the arguments in Section 7.5 carefully and explain the procedure (outlined in Section 7.6) for calculating the LIBOR/swap zero curve.

In the case of currency swaps the exchange of principal needs to be explained. Valuation methods are structurally very similar to those for interest rate swaps and can usually be covered fairly quickly.

Problems 7.19, 7.20, 7.21, 7.22, and 7.23 all work well as assignment questions. I usually ask students to hand in two of them.

QUESTIONS AND PROBLEMS

Problem 7.1.

Companies A and B have been offered the following rates per annum on a $20 million five-year loan:

	Fixed Rate	*Floating Rate*
Company A	*5.0%*	*LIBOR + 0.1%*
Company B	*6.4%*	*LIBOR + 0.6%*

Company A requires a floating-rate loan; company B requires a fixed-rate loan. Design a swap that will net a bank, acting as intermediary, 0.1% per annum and that will appear equally attractive to both companies.

A has an apparent comparative advantage in fixed-rate markets but wants to borrow floating. B has an apparent comparative advantage in floating-rate markets but wants to borrow fixed. This provides the basis for the swap. There is a 1.4% per annum differential between the fixed rates offered to the two companies and a 0.5% per annum differential between the floating rates offered to the two companies. The total gain to all parties from the swap is therefore $1.4 - 0.5 = 0.9\%$ per annum. Because the bank gets 0.1% per annum of this gain, the swap should make each of A and B 0.4% per annum better off. This means that it should lead to A borrowing at LIBOR − 0.3% and to B borrowing at 6.0%. The appropriate arrangement is therefore as shown in Figure S7.1.

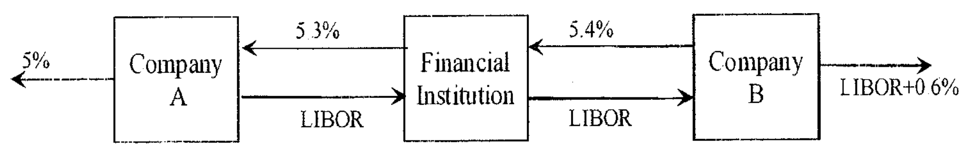


Figure S7.1 Swap for Problem 7.1

Problem 7.2.

Company X wishes to borrow U.S. dollars at a fixed rate of interest. Company Y wishes to borrow Japanese yen at a fixed rate of interest. The amounts required by the two companies are roughly the same at the current exchange rate. The companies have been quoted the following interest rates, which have been adjusted for the impact of taxes:

	Yen	*Dollars*
Company X	*5.0%*	*9.6%*
Company Y	*6.5%*	*10.0%*

Design a swap that will net a bank, acting as intermediary, 50 basis points per annum. Make the swap equally attractive to the two companies and ensure that all foreign exchange risk is assumed by the bank.

X has a comparative advantage in yen markets but wants to borrow dollars. Y has a comparative advantage in dollar markets but wants to borrow yen. This provides the basis for the swap. There is a 1.5% per annum differential between the yen rates and a 0.4% per annum differential between the dollar rates. The total gain to all parties from the swap is therefore $1.5 - 0.4 = 1.1\%$ per annum. The bank requires 0.5% per annum, leaving 0.3% per annum for each of X and Y. The swap should lead to X borrowing dollars at $9.6 - 0.3 = 9.3\%$ per annum and to Y borrowing yen at $6.5 - 0.3 = 6.2\%$ per annum. The appropriate arrangement is therefore as shown in Figure S7.2. All foreign exchange risk is borne by the bank.

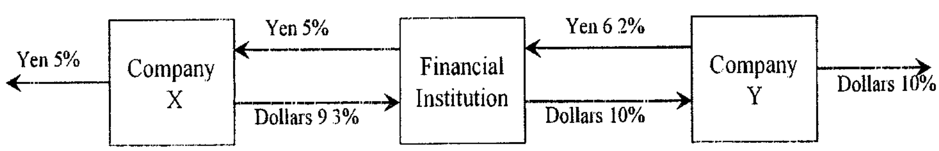


Figure S7.2 Swap for Problem 7.2

Problem 7.3.

A \$100 million interest rate swap has a remaining life of 10 months. Under the terms of the swap, six-month LIBOR is exchanged for 7% per annum (compounded semiannually). The average of the bid–offer rate being exchanged for six-month LIBOR in swaps of all maturities is currently 5% per annum with continuous compounding. The six-month LIBOR rate was 4.6% per annum two months ago. What is the current value of the swap to the party paying floating? What is its value to the party paying fixed?

In four months \$3.5 million ($= 0.5 \times 0.07 \times \$100$ million) will be received and \$2.3 million ($= 0.5 \times 0.046 \times \$100$ million) will be paid. (We ignore day count issues.) In 10 months \$3.5 million will be received, and the LIBOR rate prevailing in four months' time will be paid. The value of the fixed-rate bond underlying the swap is

$$3.5e^{-0.05\times 4/12} + 103.5e^{-0.05\times 10/12} = \$102.718 \text{ million}$$

The value of the floating-rate bond underlying the swap is

$$(100 + 2.3)e^{-0.05\times 4/12} = \$100.609 \text{ million}$$

The value of the swap to the party paying floating is $\$102.718 - \$100.609 = \$2.109$ million. The value of the swap to the party paying fixed is $-\$2.109$ million.

These results can also be derived by decomposing the swap into forward contracts. Consider the party paying floating. The first forward contract involves paying \$2.3 million and receiving \$3.5 million in four months. It has a value of $1.2e^{-0.05\times 4/12} = \$1.180$ million. To value the second forward contract, we note that the forward interest rate is 5% per annum with continuous compounding, or 5.063% per annum with semiannual compounding. The value of the forward contract is

$$100 \times (0.07 \times 0.5 - 0.05063 \times 0.5)e^{-0.05\times 10/12} = \$0.929 \text{ million}$$

The total value of the forward contracts is therefore $\$1.180 + \$0.929 = \$2.109$ million.

Problem 7.4.

Explain what a swap rate is. What is the relationship between swap rates and par yields?

A swap rate for a particular maturity is the average of the bid and offer fixed rates that a market maker is prepared to exchange for LIBOR in a standard plain vanilla swap with that maturity. The swap rate for a particular maturity is the LIBOR/swap par yield for that maturity.

Problem 7.5.

A currency swap has a remaining life of 15 months. It involves exchanging interest at 10% on £20 million for interest at 6% on \$30 million once a year. The term structure of interest rates in both the United Kingdom and the United States is currently flat, and if the swap were negotiated today the interest rates exchanged would be 4% in dollars and 7% in sterling. All interest rates are quoted with annual compounding. The current exchange rate (dollars per pound sterling) is 1.8500. What is the value of the swap to the party paying sterling? What is the value of the swap to the party paying dollars?

The swap involves exchanging the sterling interest of $20 \times 0.10 = 2.0$ million for the dollar interest of $30 \times 0.06 = \$1.8$ million. The principal amounts are also exchanged at the end of the life of the swap. The value of the sterling bond underlying the swap is

$$\frac{2}{(1.07)^{1/4}} + \frac{22}{(1.07)^{5/4}} = 22.182 \text{ million pounds}$$

The value of the dollar bond underlying the swap is

$$\frac{1.8}{(1.04)^{1/4}} + \frac{31.8}{(1.04)^{5/4}} = \$32.061 \text{ million}$$

The value of the swap to the party paying sterling is therefore

$$32.061 - (22.182 \times 1.85) = -\$8.976 \text{ million}$$

The value of the swap to the party paying dollars is +\$8.976 million. The results can also be obtained by viewing the swap as a portfolio of forward contracts. The continuously compounded interest rates in sterling and dollars are 6.766% per annum and 3.922% per annum. The 3-month and 15-month forward exchange rates are $1.85e^{(0.03922-0.06766)\times 0.25} = 1.8369$ and $1.85e^{(0.03922-0.06766)\times 1.25} = 1.7854$. The values of the two forward contracts corresponding to the exchange of interest for the party paying sterling are therefore

$$(1.8 - 2 \times 1.8369)e^{-0.03922\times 0.25} = -\$1.855 \text{ million}$$

$$(1.8 - 2 \times 1.7854)e^{-0.03922\times 1.25} = -\$1.686 \text{ million}$$

The value of the forward contract corresponding to the exchange of principals is

$$(30 - 20 \times 1.7854)e^{-0.03922\times 1.25} = -\$5.435 \text{ million}$$

The total value of the swap is $-\$1.855 - \$1.686 - \$5.435 = -\$8.976$ million.

Problem 7.6.

Explain the difference between the credit risk and the market risk in a financial contract.

Credit risk arises from the possibility of a default by the counterparty. Market risk arises from movements in market variables such as interest rates and exchange rates. A complication is that the credit risk in a swap is contingent on the values of market variables. A company's position in a swap has credit risk only when the value of the swap to the company is positive.

Problem 7.7.

A corporate treasurer tells you that he has just negotiated a five-year loan at a competitive fixed rate of interest of 5.2%. The treasurer explains that he achieved the 5.2% rate by borrowing at six-month LIBOR plus 150 basis points and swapping LIBOR for 3.7%. He goes on to say that this was possible because his company has a comparative advantage in the floating-rate market. What has the treasurer overlooked?

The rate is not truly fixed because, if the company's credit rating declines, it will not be able to roll over its floating rate borrowings at LIBOR plus 150 basis points. The effective fixed borrowing rate then increases. Suppose for example that the treasurer's spread over LIBOR increases from 150 basis points to 200 basis points. The borrowing rate increases from 5.2% to 5.7%.

Problem 7.8.

Explain why a bank is subject to credit risk when it enters into two offsetting swap contracts.

At the start of the swap, both contracts have a value of approximately zero. As time passes, it is likely that the swap values will change, so that one swap has a positive value to the bank and the other has a negative value to the bank. If the counterparty on the other side of the positive-value swap defaults, the bank still has to honor its contract with the other counterparty. It is liable to lose an amount equal to the positive value of the swap.

Problem 7.9.

Companies X and Y have been offered the following rates per annum on a $5 million 10-year investment:

	Fixed Rate	*Floating Rate*
Company X	*8.0%*	*LIBOR*
Company Y	*8.8%*	*LIBOR*

Company X requires a fixed-rate investment; company Y requires a floating-rate investment. Design a swap that will net a bank, acting as intermediary, 0.2% per annum and will appear equally attractive to X and Y.

The spread between the interest rates offered to X and Y is 0.8% per annum on fixed rate investments and 0.0% per annum on floating rate investments. This means that the total apparent benefit to all parties from the swap is 0.8% per annum. Of this 0.2% per annum will go to the bank. This leaves 0.3% per annum for each of X and Y. In other words, company X should be able to get a fixed-rate return of 8.3% per annum while company Y should be able to get a floating-rate return LIBOR + 0.3% per annum. The required swap is shown in Figure S7.3. The bank earns 0.2%, company X earns 8.3%, and company Y earns LIBOR + 0.3%.

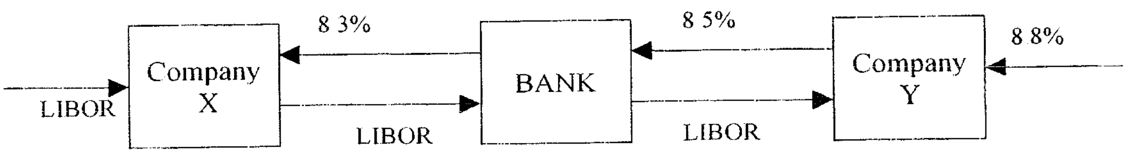


Figure S7.3 Swap for Problem 7.9

Problem 7.10.

A financial institution has entered into an interest rate swap with company X. Under the terms of the swap, it receives 10% per annum and pays six-month LIBOR on a principal of $10 million for five years. Payments are made every six months. Suppose that company X defaults on the sixth payment date (end of year 3) when the interest rate (with semiannual compounding) is 8% per annum for all maturities. What is the loss to the financial institution? Assume that six-month LIBOR was 9% per annum halfway through year 3.

At the end of year 3 the financial institution was due to receive \$500,000 (= $0.5 \times 10\%$ of \$10 million) and pay \$450,000 (= $0.5 \times 9\%$ of \$10 million). The immediate loss is therefore \$50,000. To value the remaining swap we assume than forward rates are realized. All forward rates are 8% per annum. The remaining cash flows are therefore valued on the assumption that the floating payment is $0.5 \times 0.08 \times 10,000,000 = \$400,000$ and the net payment that would be received is $500,000 - 400,000 = \$100,000$. The total cost of default is therefore the cost of foregoing the following cash flows:

year 3:	\$50,000
year $3\frac{1}{2}$:	\$100,000
year 4:	\$100,000
year $4\frac{1}{2}$:	\$100,000
year 5:	\$100,000

Discounting these cash flows to year 3 at 4% per six months we obtain the cost of the default as \$413,000.

Problem 7.11.

Companies A and B face the following interest rates (adjusted for the differential impact of taxes):

	A	*B*
U.S. dollars (floating rate)	*LIBOR + 0.5%*	*LIBOR + 1.0%*
Canadian dollars (fixed rate)	*5.0%*	*6.5%*

Assume that A wants to borrow U.S. dollars at a floating rate of interest and B wants to borrow Canadian dollars at a fixed rate of interest. A financial institution is planning to arrange a swap and requires a 50-basis-point spread. If the swap is equally attractive to A and B, what rates of interest will A and B end up paying?

Company A has a comparative advantage in the Canadian dollar fixed-rate market. Company B has a comparative advantage in the U.S. dollar floating-rate market. (This may be because of their tax positions.) However, company A wants to borrow in the U.S. dollar floating-rate market and company B wants to borrow in the Canadian dollar fixed-rate market. This gives rise to the swap opportunity.

The differential between the U.S. dollar floating rates is 0.5% per annum, and the differential between the Canadian dollar fixed rates is 1.5% per annum. The difference between the differentials is 1% per annum. The total potential gain to all parties from the swap is therefore 1% per annum, or 100 basis points. If the financial intermediary requires 50 basis points, each of A and B can be made 25 basis points better off. Thus a swap can be designed so that it provides A with U.S. dollars at LIBOR + 0.25% per annum, and B with Canadian dollars at 6.25% per annum. The swap is shown in Figure S7.4.

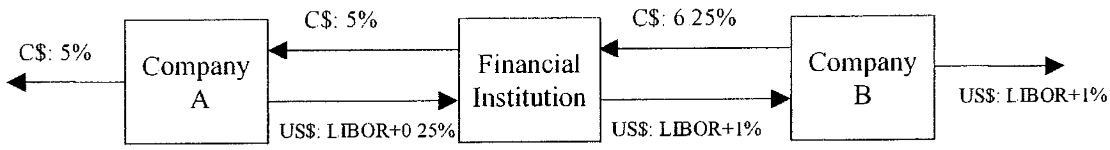


Figure S7.4 Swap for Problem 7.11

Principal payments flow in the opposite direction to the arrows at the start of the life of the swap and in the same direction as the arrows at the end of the life of the swap. The financial institution would be exposed to some foreign exchange risk which could be hedged using forward contracts.

Problem 7.12.

A financial institution has entered into a 10-year currency swap with company Y. Under the terms of the swap, the financial institution receives interest at 3% per annum in Swiss francs and pays interest at 8% per annum in U.S. dollars. Interest payments are exchanged once a year. The principal amounts are 7 million dollars and 10 million francs. Suppose that company Y declares bankruptcy at the end of year 6, when the exchange rate is $0.80 per franc. What is the cost to the financial institution? Assume that, at the end of year 6, the interest rate is 3% per annum in Swiss francs and 8% per annum in U.S. dollars for all maturities. All interest rates are quoted with annual compounding.

When interest rates are compounded annually

$$F_0 = S_0 \left(\frac{1+r}{1+r_f} \right)^T$$

where F_0 is the T-year forward rate, S_0 is the spot rate, r is the domestic risk-free rate, and r_f is the foreign risk-free rate. As $r = 0.08$ and $r_f = 0.03$, the spot and forward exchange rates at the end of year 6 are

spot:	0.8000
1 year forward:	0.8388
2 year forward:	0.8796
3 year forward:	0.9223
4 year forward:	0.9670

The value of the swap at the time of the default can be calculated on the assumption that forward rates are realized. The cash flows lost as a result of the default are therefore as follows:

Year	Dollar Paid	Swiss Franc Received	Forward Rate	Dollar Equivalent of Swiss Franc Received	Cash Flow Lost
6	560,000	300,000	0.8000	240,000	(320,000)
7	560,000	300,000	0.8388	251,600	(308,400)
8	560,000	300,000	0.8796	263,900	(296,100)
9	560,000	300,000	0.9223	276,700	(283,300)
10	7,560,000	10,300,000	0.9670	9,960,100	2,400,100

Discounting the numbers in the final column to the end of year 6 at 8% per annum, the cost of the default is $679,800.

Note that, if this were the only contract entered into by company Y, it would make no sense for the company to default at the end of year six as the exchange of payments at that time has a positive value to company Y. In practice company Y is likely to be defaulting and declaring bankruptcy for reasons unrelated to this particular contract and payments on the contract are likely to stop when bankruptcy is declared.

Problem 7.13.

After it hedges its foreign exchange risk using forward contracts, is the financial institution's average spread in Figure 7.10 likely to be greater than or less than 20 basis points? Explain your answer.

The financial institution will have to buy 1.1% of the AUD principal in the forward market for each year of the life of the swap. Since AUD interest rates are higher than dollar interest rates, AUD is at a discount in forward markets. This means that the AUD purchased for year 2 is less expensive than that purchased for year 1; the AUD purchased for year 3 is less expensive than that purchased for year 2; and so on. This works in favor of the financial institution and means that its spread increases with time. The spread is always above 20 basis points.

Problem 7.14.

"Companies with high credit risks are the ones that cannot access fixed-rate markets directly. They are the companies that are most likely to be paying fixed and receiving floating in an interest rate swap." Assume that this statement is true. Do you think it increases or decreases the risk of a financial institution's swap portfolio? Assume that companies are most likely to default when interest rates are high.

Consider a plain-vanilla interest rate swap involving two companies X and Y. We suppose that X is paying fixed and receiving floating while Y is paying floating and receiving fixed.

The quote suggests that company X will usually be less creditworthy than company Y. (Company X might be a BBB-rated company that has difficulty in accessing fixed-rate markets directly; company Y might be a AAA-rated company that has no difficulty accessing fixed or floating rate markets.) Presumably company X wants fixed-rate funds and company Y wants floating-rate funds.

The financial institution will realize a loss if company Y defaults when rates are high or if company X defaults when rates are low. These events are relatively unlikely since (a)

Y is unlikely to default in any circumstances and (b) defaults are less likely to happen when rates are low. For the purposes of illustration, suppose that the probabilities of various events are as follows:

Default by Y:	0.001
Default by X:	0.010
Rates high when default occurs:	0.7
Rates low when default occurs:	0.3

The probability of a loss is

$$0.001 \times 0.7 + 0.010 \times 0.3 = 0.0037$$

If the roles of X and Y in the swap had been reversed the probability of a loss would be

$$0.001 \times 0.3 + 0.010 \times 0.7 = 0.0073$$

Assuming companies are more likely to default when interest rates are high, the above argument shows that the observation in quotes has the effect of decreasing the risk of a financial institution's swap portfolio. It is worth noting that the assumption that defaults are more likely when interest rates are high is open to question. The assumption is motivated by the thought that high interest rates often lead to financial difficulties for corporations. However, there is often a time lag between interest rates being high and the resultant default. When the default actually happens interest rates may be relatively low.

Problem 7.15.

Why is the expected loss from a default on a swap less than the expected loss from the default on a loan with the same principal?

In an interest-rate swap a financial institution's exposure depends on the difference between a fixed-rate of interest and a floating-rate of interest. It has no exposure to the notional principal. In a loan the whole principal can be lost.

Problem 7.16.

A bank finds that its assets are not matched with its liabilities. It is taking floating-rate deposits and making fixed-rate loans. How can swaps be used to offset the risk?

The bank is paying a floating-rate on the deposits and receiving a fixed-rate on the loans. It can offset its risk by entering into interest rate swaps (with other financial institutions or corporations) in which it contracts to pay fixed and receive floating.

Problem 7.17.

Explain how you would value a swap that is the exchange of a floating rate in one currency for a fixed rate in another currency.

The floating payments can be valued in currency A by (i) assuming that the forward rates are realized, and (ii) discounting the resulting cash flows at appropriate currency A discount rates. Suppose that the value is V_A. The fixed payments can be valued in currency B by discounting them at the appropriate currency B discount rates. Suppose that the value is V_B. If Q is the current exchange rate (number of units of currency A per unit of currency B), the value of the swap in currency A is $V_A - QV_B$. Alternatively, it is $V_A/Q - V_B$ in currency B.

Problem 7.18.

The LIBOR zero curve is flat at 5% (continuously compounded) out to 1.5 years. Swap rates for 2- and 3-year semiannual pay swaps are 5.4% and 5.6%, respectively. Estimate the LIBOR zero rates for maturities of 2.0, 2.5, and 3.0 years. (Assume that the 2.5-year swap rate is the average of the 2- and 3-year swap rates.)

The two-year swap rate is 5.4%. This means that a two-year LIBOR bond paying a semiannual coupon at the rate of 5.4% per annum sells for par. If R_2 is the two-year LIBOR zero rate

$$2.7e^{-0.05\times 0.5} + 2.7e^{-0.05\times 1.0} + 2.7e^{-0.05\times 1.5} + 102.7e^{-R_2\times 2.0} = 100$$

Solving this gives $R_2 = 0.05342$. The 2.5-year swap rate is assumed to be 5.5%. This means that a 2.5-year LIBOR bond paying a semiannual coupon at the rate of 5.5% per annum sells for par. If $R_{2.5}$ is the 2.5-year LIBOR zero rate

$$2.75e^{-0.05\times 0.5} + 2.75e^{-0.05\times 1.0} + 2.75e^{-0.05\times 1.5} + 2.75e^{-0.05342\times 2.0} + 102.75e^{-R_{2.5}\times 2.5} = 100$$

Solving this gives $R_{2.5} = 0.05442$. The 3-year swap rate is 5.6%. This means that a 3-year LIBOR bond paying a semiannual coupon at the rate of 5.6% per annum sells for par. If R_3 is the three-year LIBOR zero rate

$$2.8e^{-0.05\times 0.5} + 2.8e^{-0.05\times 1.0} + 2.8e^{-0.05\times 1.5} + 2.8e^{-0.05342\times 2.0} + 2.8e^{-0.05442\times 2.5}$$
$$+102.8e^{-R_3\times 3.0} = 100$$

Solving this gives $R_3 = 0.05544$. The zero rates for maturities 2.0, 2.5, and 3.0 years are therefore 5.342%, 5.442%, and 5.544%, respectively.

ASSIGNMENT QUESTIONS

Problem 7.19.

The one-year LIBOR rate is 10%. A bank trades swaps where a fixed rate of interest is exchanged for 12-month LIBOR with payments being exchanged annually. Two- and three-year swap rates (expressed with annual compounding) are 11% and 12% per annum. Estimate the two- and three-year LIBOR zero rates.

The two-year swap rate implies that a two-year LIBOR bond with a coupon of 11% sells for par. If R_2 is the two-year zero rate

$$11e^{-0.10\times 1.0} + 111e^{-R_2\times 2.0} = 100$$

so that $R_2 = 0.1046$ The three-year swap rate implies that a three-year LIBOR bond with a coupon of 12% sells for par. If R_3 is the three-year zero rate

$$12e^{-0.10\times1.0} + 12e^{-0.1046\times2.0} + 112e^{-R_3\times3.0} = 100$$

so that $R_3 = 0.1146$ The two- and three-year rates are therefore 10.46% and 11.46% with continuous compounding.

Problem 7.20.

Company A, a British manufacturer, wishes to borrow U.S. dollars at a fixed rate of interest. Company B, a U.S. multinational, wishes to borrow sterling at a fixed rate of interest. They have been quoted the following rates per annum (adjusted for differential tax effects):

	Sterling	*U.S. Dollars*
Company A	*11.0%*	*7.0%*
Company B	*10.6%*	*6.2%*

Design a swap that will net a bank, acting as intermediary, 10 basis points per annum and that will produce a gain of 15 basis points per annum for each of the two companies.

The spread between the interest rates offered to A and B is 0.4% (or 40 basis points) on sterling loans and 0.8% (or 80 basis points) on U.S. dollar loans. The total benefit to all parties from the swap is therefore

$$80 - 40 = 40 \text{ basis points}$$

It is therefore possible to design a swap which will earn 10 basis points for the bank while making each of A and B 15 basis points better off than they would be by going directly to financial markets. One possible swap is shown in Figure M7.1. Company A borrows at an effective rate of 6.85% per annum in U.S. dollars.

Company B borrows at an effective rate of 10.45% per annum in sterling. The bank earns a 10-basis-point spread. The way in which currency swaps such as this operate is as follows. Principal amounts in dollars and sterling that are roughly equivalent are chosen. These principal amounts flow in the opposite direction to the arrows at the time the swap is initiated. Interest payments then flow in the same direction as the arrows during the life of the swap and the principal amounts flow in the same direction as the arrows at the end of the life of the swap.

Note that the bank is exposed to some exchange rate risk in the swap. It earns 65 basis points in U.S. dollars and pays 55 basis points in sterling. This exchange rate risk could be hedged using forward contracts.

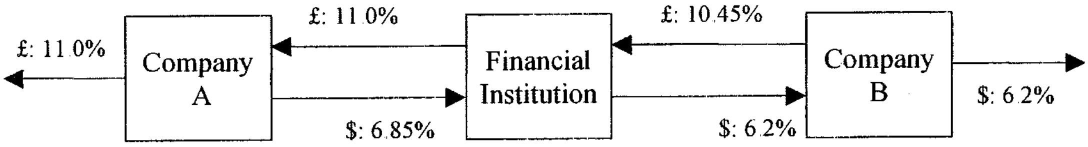


Figure M7.1 One Possible Swap for Problem 7.20

Problem 7.21.

Under the terms of an interest rate swap, a financial institution has agreed to pay 10% per annum and to receive three-month LIBOR in return on a notional principal of $100 million with payments being exchanged every three months. The swap has a remaining life of 14 months. The average of the bid and offer fixed rates currently being swapped for three-month LIBOR is 12% per annum for all maturities. The three-month LIBOR rate one month ago was 11.8% per annum. All rates are compounded quarterly. What is the value of the swap?

The swap can be regarded as a long position in a floating-rate bond combined with a short position in a fixed-rate bond. The correct discount rate is 12% per annum with quarterly compounding or 11.82% per annum with continuous compounding.

Immediately after the next payment the floating-rate bond will be worth $100 million. The next floating payment ($ million) is

$$0.118 \times 100 \times 0.25 = 2.95$$

The value of the floating-rate bond is therefore

$$102.95e^{-0.1182\times 2/12} = 100.941$$

The value of the fixed-rate bond is

$$2.5e^{-0.1182\times 2/12} + 2.5e^{-0.1182\times 5/12} + 2.5e^{-0.1182\times 8/12}$$

$$+2.5e^{-0.1182\times 11/12} + 102.5e^{-0.1182\times 14/12} = 98.678$$

The value of the swap is therefore

$$100.941 - 98.678 = \$2.263 \text{ million}$$

As an alternative approach we can value the swap as a series of forward rate agreements. The calculated value is

$$(2.95 - 2.5)e^{-0.1182\times 2/12} + (3.0 - 2.5)e^{-0.1182\times 5/12}$$

$$+(3.0 - 2.5)e^{0.1182\times 8/12} + (3.0 - 2.5)e^{-0.1182\times 11/12}$$

$$+(3.0 - 2.5)e^{-0.1182\times 14/12} = \$2.263 \text{ million}$$

which is in agreement with the answer obtained using the first approach.

Problem 7.22.

Suppose that the term structure of interest rates is flat in the United States and Australia. The USD interest rate is 7% per annum and the AUD rate is 9% per annum. The current value of the AUD is 0.62 USD. Under the terms of a swap agreement, a financial institution pays 8% per annum in AUD and receives 4% per annum in USD. The principals in the two currencies are $12 million USD and 20 million AUD. Payments are exchanged every year, with one exchange having just taken place. The swap will last two more years. What is the value of the swap to the financial institution? Assume all interest rates are continuously compounded.

The financial institution is long a dollar bond and short a USD bond. The value of the dollar bond (in millions of dollars) is

$$0.48e^{-0.07\times 1} + 12.48e^{-0.07\times 2} = 11.297$$

The value of the AUD bond (in millions of AUD) is

$$1.6e^{-0.09\times 1} + 21.6e^{-0.09\times 2} = 19.504$$

The value of the swap (in millions of dollars) is therefore

$$11.297 - 19.504 \times 0.62 = -0.795$$

or −$795,000.

As an alternative we can value the swap as a series of forward foreign exchange contracts. The one-year forward exchange rate is $0.62e^{-0.02} = 0.6077$. The two-year forward exchange rate is $0.62e^{-0.02\times 2} = 0.5957$. The value of the swap in millions of dollars is therefore

$$(0.48 - 1.6 \times 0.6077)e^{-0.07\times 1} + (12.48 - 21.6 \times 0.5957)e^{-0.07\times 2} = -0.795$$

which is in agreement with the first calculation.

Problem 7.23.

Company X is based in the United Kingdom and would like to borrow $50 million at a fixed rate of interest for five years in U.S. funds. Because the company is not well known in the United States, this has proved to be impossible. However, the company has been quoted 12% per annum on fixed-rate five-year sterling funds. Company Y is based in the United States and would like to borrow the equivalent of $50 million in sterling funds for five years at a fixed rate of interest. It has been unable to get a quote but has been offered U.S. dollar funds at 10.5% per annum. Five-year government bonds currently yield 9.5% per annum in the United States and 10.5% in the United Kingdom. Suggest an appropriate currency swap that will net the financial intermediary 0.5% per annum.

There is a 1% differential between the yield on sterling and dollar 5-year bonds. The financial intermediary could use this differential when designing a swap. For example, it could (a) allow company X to borrow dollars at 1% per annum less than the rate offered on sterling funds, that is, at 11% per annum and (b) allow company Y to borrow sterling at 1% per annum more than the rate offered on dollar funds, that is, at $11\frac{1}{2}$% per annum. However, as shown in Figure M7.2, the financial intermediary would not then earn a positive spread.

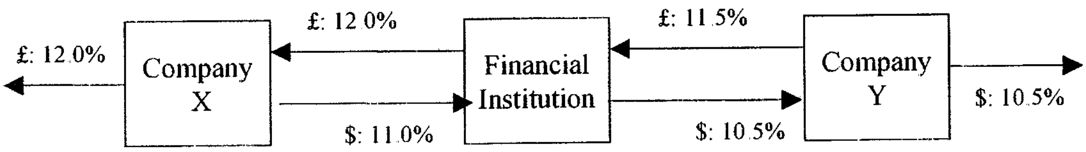


Figure M7.2 First attempt at designing swap for Problem 7.23

To make 0.5% per annum, the financial intermediary could add 0.25% per annum, to the rates paid by each of X and Y. This means that X pays 11.25% per annum, for dollars and Y pays 11.75% per annum, for sterling and leads to the swap shown in Figure M7.3. The financial intermediary would be exposed to some foreign exchange risk in this swap. This could be hedged using forward contracts.

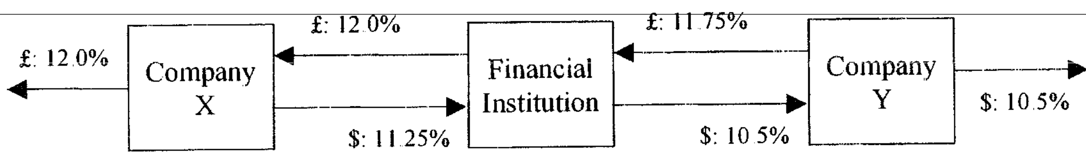


Figure M7.3 Final Swap for Problem 7.23

CHAPTER 8
Mechanics of Options Markets

Notes for the Instructor

This chapter provides information on how options markets work. I usually go through the chapter fairly quickly leaving students to read the details for themselves. Points I spend time on are the payoffs from the four option positions and how the terms of options change when there are dividends and stock splits. There is less material on employee stock options in this chapter than before. This is because there is now a whole chapter (Chapter 14) devoted to this topic.

Problems 8.24 and 8.26 work well for class discussion. Problems 8.23 and 8.25 can be used as assignment questions.

QUESTIONS AND PROBLEMS

Problem 8.1.

An investor buys a European put on a share for \$3. The stock price is \$42 and the strike price is \$40. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

The investor makes a profit if the price of the stock on the expiration date is less than \$37. In these circumstances the gain from exercising the option is greater than \$3. The option will be exercised if the stock price is less than \$40 at the maturity of the option. The variation of the investor's profit with the stock price in Figure S8.1.

Problem 8.2.

An investor sells a European call on a share for \$4. The stock price is \$47 and the strike price is \$50. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

The investor makes a profit if the price of the stock is below \$54 on the expiration date. If the stock price is below \$50, the option will not be exercised, and the investor makes a profit of \$4. If the stock price is between \$50 and \$54, the option is exercised and the investor makes a profit between \$0 and \$4. The variation of the investor's profit with the stock price is as shown in Figure S8.2.

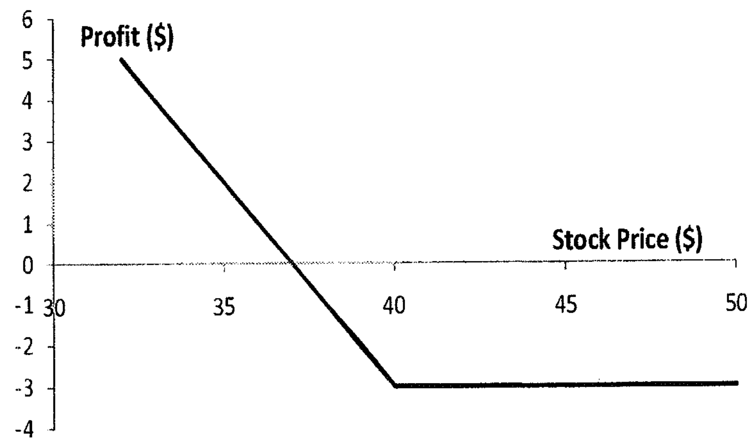


Figure S8.1 Investor's profit in Problem 8.1

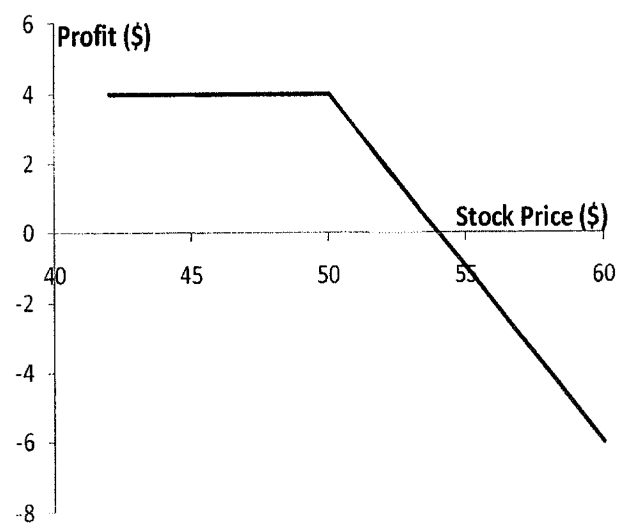


Figure S8.2 Investor's profit in Problem 8.2

Problem 8.3.

An investor sells a European call option with strike price of K and maturity T and buys a put with the same strike price and maturity. Describe the investor's position.

The payoff to the investor is

$$-\max(S_T - K, 0) + \max(K - S_T, 0)$$

This is $K - S_T$ in all circumstances. The investor's position is the same as a short position in a forward contract with delivery price K.

Problem 8.4.

Explain why brokers require margins when clients write options but not when they buy options.

When an investor buys an option, cash must be paid up front. There is no possibility of future liabilities and therefore no need for a margin account. When an investor sells an option, there are potential future liabilities. To protect against the risk of a default, margins are required.

Problem 8.5.

A stock option is on a February, May, August, and November cycle. What options trade on (a) April 1 and (b) May 30?

On April 1 options trade with expiration months of April, May, August, and November. On May 30 options trade with expiration months of June, July, August, and November.

Problem 8.6.

A company declares a 2-for-1 stock split. Explain how the terms change for a call option with a strike price of $60.

The strike price is reduced to $30, and the option gives the holder the right to purchase twice as many shares.

Problem 8.7.

"Employee stock options issued by a company are different from regular exchange-traded call options on the company's stock because they can affect the capital structure of the company." Explain this statement.

The exercise of employee stock options usually leads to new shares being issued by the company and sold to the employee. This changes the amount of equity in the capital structure. When a regular exchange-traded option is exercised no new shares are issued and the company's capital structure is not affected.

Problem 8.8.

A corporate treasurer is designing a hedging program involving foreign currency options. What are the pros and cons of using (a) the Philadelphia Stock Exchange and (b) the over-the-counter market for trading?

The Philadelphia Exchange offers European and American options with standard strike prices and times to maturity. Options in the over-the-counter market have the advantage that they can be tailored to meet the precise needs of the treasurer. Their disadvantage is that they expose the treasurer to some credit risk. Exchanges organize their trading so that there is virtually no credit risk.

Problem 8.9.

Suppose that a European call option to buy a share for \$100.00 costs \$5.00 and is held until maturity. Under what circumstances will the holder of the option make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a long position in the option depends on the stock price at maturity of the option.

Ignoring the time value of money, the holder of the option will make a profit if the stock price at maturity of the option is greater than \$105. This is because the payoff to the holder of the option is, in these circumstances, greater than the \$5 paid for the option. The option will be exercised if the stock price at maturity is greater than \$100. Note that if the stock price is between \$100 and \$105 the option is exercised, but the holder of the option takes a loss overall. The profit from a long position is as shown in Figure S8.3.

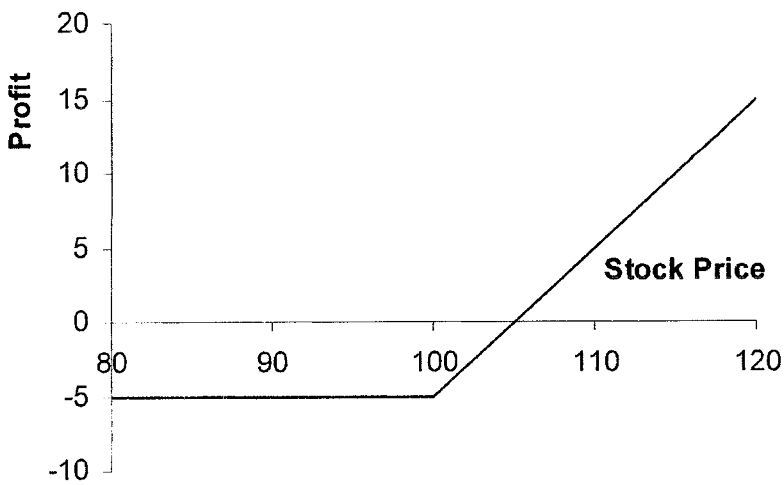


Figure S8.3 Profit from long position in Problem 8.9

Problem 8.10.

Suppose that a European put option to sell a share for \$60 costs \$8 and is held until maturity. Under what circumstances will the seller of the option (the party with the short position) make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a short position in the option depends on the stock price at maturity of the option.

Ignoring the time value of money, the seller of the option will make a profit if the stock price at maturity is greater than \$52.00. This is because the cost to the seller of the option is in these circumstances less than the price received for the option. The option will be exercised if the stock price at maturity is less than \$60.00. Note that if the stock price is between \$52.00 and \$60.00 the seller of the option makes a profit even though the option is exercised. The profit from the short position is as shown in Figure S8.4.

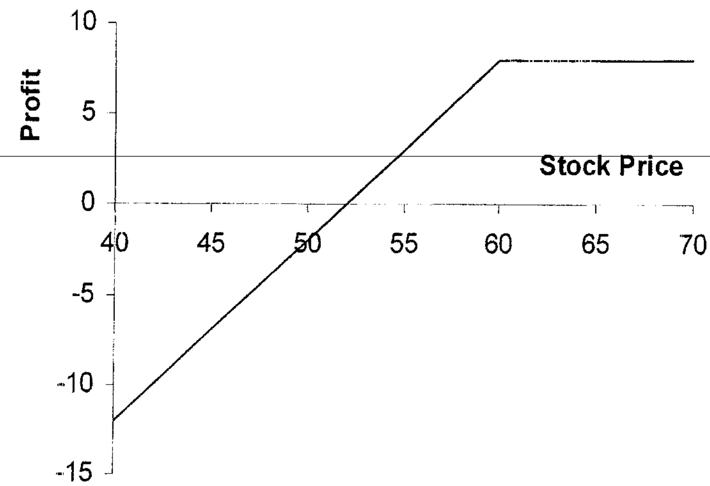


Figure S8.4 Profit from short position in Problem 8.10

Problem 8.11.

Describe the terminal value of the following portfolio: a newly entered-into long forward contract on an asset and a long position in a European put option on the asset with the same maturity as the forward contract and a strike price that is equal to the forward price of the asset at the time the portfolio is set up. Show that the European put option has the same value as a European call option with the same strike price and maturity.

The terminal value of the long forward contract is:

$$S_T - F_0$$

where S_T is the price of the asset at maturity and F_0 is the forward price of the asset at the time the portfolio is set up. (The delivery price in the forward contract is also F_0.)

The terminal value of the put option is:

$$\max(F_0 - S_T,\ 0)$$

The terminal value of the portfolio is therefore

$$S_T - F_0 + \max(F_0 - S_T,\ 0)$$

$$= \max(0,\ S_T - F_0]$$

This is the same as the terminal value of a European call option with the same maturity as the forward contract and an exercise price equal to F_0.

We have shown that the forward contract plus the put is worth the same as a call with the same strike price and time to maturity as the put. The forward contract is worth zero at the time the portfolio is set up. It follows that the put is worth the same as the call at the time the portfolio is set up.

Problem 8.12.

A trader buys a call option with a strike price of \$45 and a put option with a strike price of \$40. Both options have the same maturity. The call costs \$3 and the put costs \$4. Draw a diagram showing the variation of the trader's profit with the asset price.

Figure S8.5 shows the variation of the trader's position with the asset price. We can divide the alternative asset prices into three ranges:

(a) When the asset price less than \$40, the put option provides a payoff of $40 - S_T$ and the call option provides no payoff. The options cost \$7 and so the total profit is $33 - S_T$.

(b) When the asset price is between \$40 and \$45, neither option provides a payoff. There is a net loss of \$7.

(c) When the asset price greater than \$45, the call option provides a payoff of $S_T - 45$ and the put option provides no payoff. Taking into account the \$7 cost of the options, the total profit is $S_T - 52$.

The trader makes a profit (ignoring the time value of money) if the stock price is less than \$33 or greater than \$52. This type of trading strategy is known as a strangle and is discussed in Chapter 10.

Problem 8.13.

Explain why an American option is always worth at least as much as a European option on the same asset with the same strike price and exercise date.

The holder of an American option has all the same rights as the holder of a European option and more. It must therefore be worth at least as much. If it were not, an arbitrageur could short the European option and take a long position in the American option.

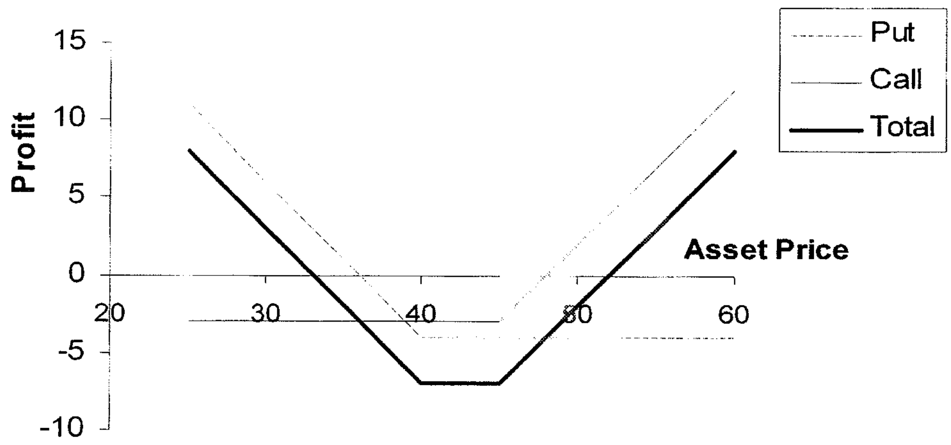


Figure S8.5 Profit from trading strategy in Problem 8.12

Problem 8.14.

Explain why an American option is always worth at least as much as its intrinsic value.

The holder of an American option has the right to exercise it immediately. The American option must therefore be worth at least as much as its intrinsic value. If it were not an arbitrageur could lock in a sure profit by buying the option and exercising it immediately.

Problem 8.15.

Explain carefully the difference between writing a put option and buying a call option.

Writing a put gives a payoff of $\min(S_T - K, 0)$. Buying a call gives a payoff of $\max(S_T - K, 0)$. In both cases the potential payoff is $S_T - K$. The difference is that for a written put the counterparty chooses whether you get the payoff (and will allow you to get it only when it is negative to you). For a long call you decide whether you get the payoff (and you choose to get it when it is positive to you.)

Problem 8.16.

The treasurer of a corporation is trying to choose between options and forward contracts to hedge the corporation's foreign exchange risk. Discuss the advantages and disadvantages of each.

Forward contracts lock in the exchange rate that will apply to a particular transaction in the future. Options provide insurance that the exchange rate will not be worse than some level. The advantage of a forward contract is that uncertainty is eliminated as far as possible. The disadvantage is that the outcome with hedging can be significantly worse than the outcome with no hedging. This disadvantage is not as marked with options. However, unlike forward contracts, options involve an up-front cost.

Problem 8.17.

Consider an exchange-traded call option contract to buy 500 shares with a strike price of $40 and maturity in four months. Explain how the terms of the option contract change when there is

a. A 10% stock dividend
b. A 10% cash dividend
c. A 4-for-1 stock split

(a) The option contract becomes one to buy $500 \times 1.1 = 550$ shares with an exercise price $40/1.1 = 36.36$.

(b) There is no effect. The terms of an options contract are not normally adjusted for cash dividends.

(c) The option contract becomes one to buy $500 \times 4 = 2,000$ shares with an exercise price of $40/4 = \$10$.

Problem 8.18.

"If most of the call options on a stock are in the money, it is likely that the stock price has risen rapidly in the last few months." Discuss this statement.

The exchange has certain rules governing when trading in a new option is initiated. These mean that the option is close-to-the-money when it is first traded. If all call options are in the money it is therefore likely that the stock price has risen since trading in the option began.

Problem 8.19.

What is the effect of an unexpected cash dividend on (a) a call option price and (b) a put option price?

An unexpected cash dividend would reduce the stock price on the ex-dividend date. This stock price reduction would not be anticipated by option holders. As a result there would be a reduction in the value of a call option and an increase the value of a put option. (Note that the terms of an option are adjusted for cash dividends only in exceptional circumstances.)

Problem 8.20.

Options on General Motors stock are on a March, June, September, and December cycle. What options trade on (a) March 1, (b) June 30, and (c) August 5?

(a) March, April, June and September

(b) July, August, September, December

(c) August, September, December, March.
Longer dated options may also trade.

Problem 8.21.

Explain why the market maker's bid-offer spread represents a real cost to options investors.

A "fair" price for the option can reasonably be assumed to be half way between the bid and the offer price quoted by a market maker. An investor typically buys at the market maker's offer and sells at the market maker's bid. Each time he or she does this there is a hidden cost equal to half the bid-offer spread.

Problem 8.22.

A United States investor writes five naked call option contracts. The option price is \$3.50, the strike price is \$60.00, and the stock price is \$57.00. What is the initial margin requirement?

The two calculations are necessary to determine the initial margin. The first gives

$$500 \times (3.5 + 0.2 \times 57 - 3) = 5,950$$

The second gives

$$500 \times (3.5 + 0.1 \times 57) = 4,600$$

The initial margin is the greater of these, or \$5,950. Part of this can be provided by the initial amount of $500 \times 3.5 = \$1,750$ received for the options.

ASSIGNMENT QUESTIONS

Problem 8.23.

The price of a stock is \$40. The price of a one-year European put option on the stock with a strike price of \$30 is quoted as \$7 and the price of a one-year European call option on the stock with a strike price of \$50 is quoted as \$5. Suppose that an investor buys 100 shares, shorts 100 call options, and buys 100 put options. Draw a diagram illustrating how the investor's profit or loss varies with the stock price over the next year. How does your answer change if the investor buys 100 shares, shorts 200 call options, and buys 200 put options?

Figure M8.1 shows the way in which the investor's profit varies with the stock price in the first case. For stock prices less than \$30 there is a loss of \$1,200. As the stock price increases from \$30 to \$50 the profit increases from –\$1,200 to \$800. Above \$50 the profit is \$800. Students may express surprise that a call which is \$10 out of the money is less expensive than a put which is \$10 out of the money. This could be because of dividends or the crashophobia phenomenon discussed in Chapter 18.

Figure M8.2 shows the way in which the profit varies with stock price in the second case. In this case the profit pattern has a zigzag shape. The problem illustrates how many

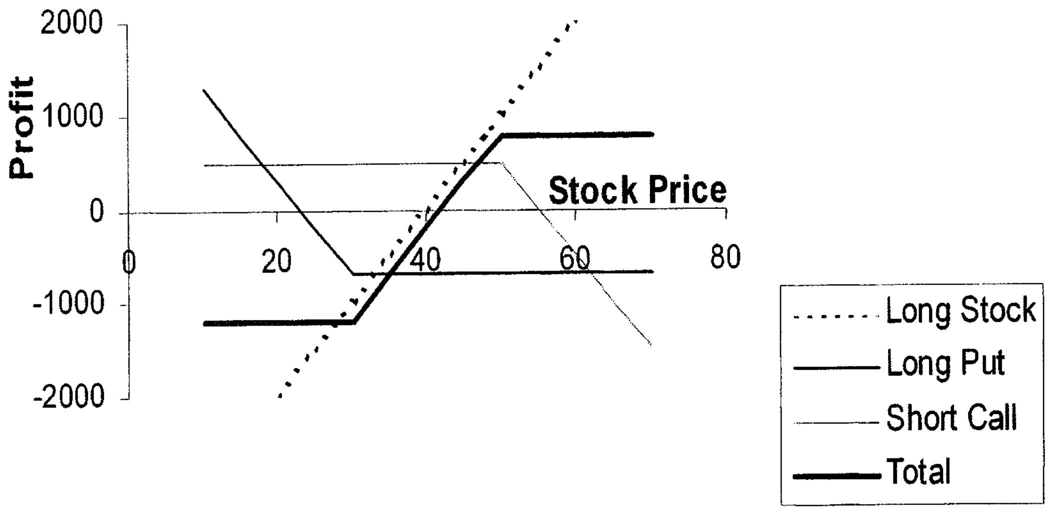


Figure M8.1 Profit in first case considered Problem 8.25

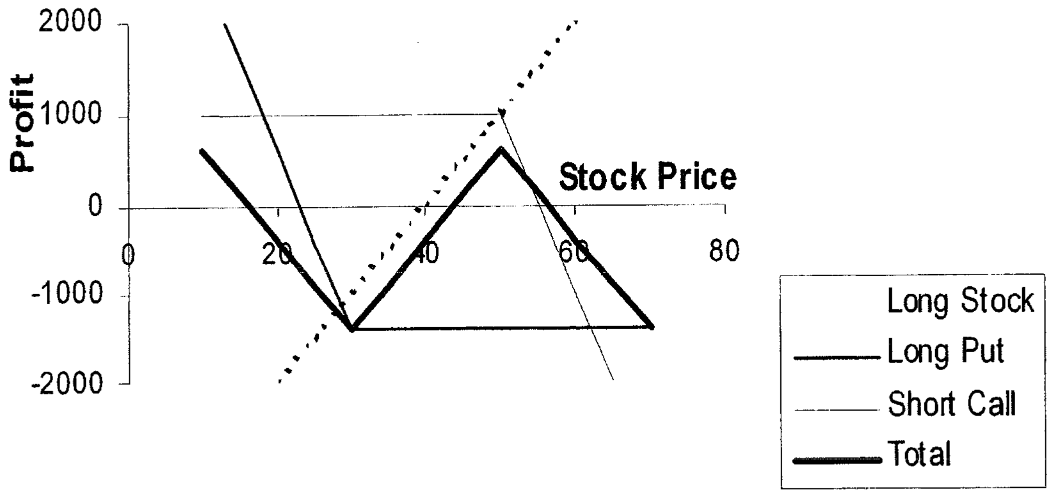


Figure M8.2 Profit for the second case considered Problem 8.25

different patterns can be obtained by including calls, puts, and the underlying asset in a portfolio.

Problem 8.24.

"If a company does not do better than its competitors but the stock market goes

up, executives do very well from their stock options. This makes no sense" Discuss this viewpoint. Can you think of alternatives to the usual executive stock option plan that take the viewpoint into account.

Executive stock option plans account for a high percentage of the total remuneration received by executives. When the market is rising fast (as it was for much of the 1990s) many corporate executives do very well out of their stock option plans — even when their company does worse than its competitors. Large institutional investors have argued that executive stock options should be structured so that the payoff depends how the company has performed relative to an appropriate industry index. In a regular executive stock option the strike price is the stock price at the time the option is issued. In the type of relative-performance stock option favored by institutional investors, the strike price at time t is $S_0 I_t / I_0$ where S_0 is the company's stock price at the time the option is issued, I_0 is the value of an equity index for the industry in which the company operates at the time the option is issued, and I_t is the value of the index at time t. If the company's performance equals the performance of the industry, the options are always at-the-money. If the company outperforms the industry, the options become in the money. If the company underperforms the industry, the options become out of the money. Note that a relative performance stock option can provide a payoff when both the market and the company's stock price decline.

Relative performance stock options clearly provide a better way of rewarding senior management for superior performance. Some companies have argued that, if they introduce relative performance options when their competitors do not, they will lose some of their top management talent.

Problem 8.25.

Use DerivaGem to calculate the value of an American put option on a nondividend paying stock when the stock price is \$30, the strike price is \$32, the risk-free rate is 5%, the volatility is 30%, and the time to maturity is 1.5 years. (Choose "Binomial American" for the "option type" and 50 time steps.)

a. *What is the option's intrinsic value?*
b. *What is the option's time value?*
c. *What would a time value of zero indicate? What is the value of an option with zero time value?*
d. *Using a trial and error approach calculate how low the stock price would have to be for the time value of the option to be zero.*

DerivaGem shows that the value of the option is 4.57. The option's intrinsic value is $32 - 30 = 2.00$. The option's time value is therefore $4.57 - 2.00 = 2.57$. A time value of zero would indicate that it is optimal to exercise the option immediately. In this case the value of the option would equal its intrinsic value. When the stock price is 20, DerivaGem gives the value of the option as 12, which is its intrinsic value. When the stock price is 25, DerivaGem gives the value of the options as 7.54, indicating that the time value is still positive (= 0.54). Keeping the number of time steps equal to 50, trial and error indicates the time value disappears when the stock price is reduced to 21.69 or lower. (With 500 time steps this estimate of how low the stock price must become is reduced to 21.35.)

Problem 8.26.

On July 20, 2004 Microsoft surprised the market by announcing a \$3 dividend. The ex-dividend date was November 17, 2004 and the payment date was December 2, 2004. Its stock price at the time was about \$28. It also changed the terms of its employee stock options so that each exercise price was adjusted downward to

$$\text{Pre-dividend Exercise Price} \times \frac{\text{Closing Price}-\$3.00}{\text{Closing Price}}$$

The number of shares covered by each stock option outstanding was adjusted upward to

$$\text{Number of Shares Pre-dividend} \times \frac{\text{Closing Price}}{\text{Closing Price}-\$3.00}$$

"Closing Price" means the official NASDAQ closing price of a share of Microsoft common stock on the last trading day before the ex-dividend date.

Evaluate this adjustment. Compare it with the system used by exchanges to adjust for extraordinary dividends (see Business Snapshot 8.1).

Suppose that the closing stock price is \$28 and an employee has 1000 options with a strike price of \$24. Microsoft's adjustment involves changing the strike price to $24 \times 25/28 = 21.4286$ and changing the number of options to $1000 \times 28/25 = 1,120$. The system used by exchanges would involve keeping the number of options the same and reducing the strike price by \$3 to \$21.

The Microsoft adjustment is more complicated than that used by the exchange because it requires a knowledge of the Microsoft's stock price immediately before the stock goes ex-dividend. However, arguably it is a better adjustment than the one used by the exchange. Before the adjustment the employee has the right to pay \$24,000 for Microsoft stock that is worth \$28,000. After the adjustment the employee also has the option to pay \$24,000 for Microsoft stock worth \$28,000. Under the adjustment rule used by exchanges the employee would have the right to buy stock worth \$25,000 for \$21,000. If the volatility of Microsoft remains the same this is a less valuable option.

One complication here is that Microsoft's volatility does not remain the same. It can be expected to go up because some cash (a zero risk asset) has been transferred to shareholders. The employees therefore have the same basic option as before but the volatility of Microsoft can be expected to increase. The employees are slightly better off because the value of an option increases with volatility.

CHAPTER 9

Properties of Stock Options

Notes for the Instructor

This chapter outlines a number of relationships between a stock option price and the underlying stock price that do not involve any assumptions about the volatility of the stock's price. As will be evident from the slides, I like to present students with sets of numerical data for c, C, p, P, S_0, K, r, T, and D that violate the relationships in the chapter and ask them what trades they would do. This usually results in good classroom interaction.

I devote most time in class to

1. The $c \geq S_0 - Ke^{-rT}$ result
2. The early exercise arguments for American calls
3. The put–call parity result for European options

When discussing the early exercise of American calls on non-dividend-paying stocks I present students with a situation where the option is deep-in-the-money ($S_0 = 100$; $T = 0.25$; $K = 60$) and ask whether they would exercise early when (a) they want to keep the stock as part of their portfolio and (b) when they think the stock is a "dog". In the first case, they should delay exercise and thereby delay paying the strike price. In the second case they should sell the option to an investor who does want to keep the stock as part of his or her portfolio. (Such an investor must exist as otherwise the stock price would not be \$100). This investor will pay at least 100 minus the present value of 60 for the option. So, if the possibility of the stock price falling below \$60 is ignored, the option should not be exercised early. When the possibility of the stock price falling below \$60 is recognized, we become even less inclined to exercise early.

Two follow-up questions are "Why does this argument not work for put options?" and "Why are employee stock options frequently exercised early?" The answer to the first question is that the strike price is paid not received in the case of put options and so the time value of money argument does not work. The answer to the second question is that employee stock options cannot be traded.

This can be a good time to introduce the Excel-based software DerivaGem that goes with the book (if the instructor has not already done so). The software can be used to plot the relationship between call/put prices and the variables: S_0, K, r, σ, and T. It can also be used to check put–call parity and investigate the difference between American and European option prices.

As already mentioned I often use Problem 9.20 in class. Problem 9.22 is a short assignment question. Problems 9.23, 9.24, and 9.25 are more challenging and can be used for assignments or class discussion. Problem 9.26 gets students started with the DerivaGem software.

QUESTIONS AND PROBLEMS

Problem 9.1.

List the six factors affecting stock option prices.

The six factors affecting stock option prices are the stock price, strike price, risk-free interest rate, volatility, time to maturity, and dividends.

Problem 9.2.

What is a lower bound for the price of a four-month call option on a non-dividend-paying stock when the stock price is $28, the strike price is $25, and the risk-free interest rate is 8% per annum?

The lower bound is

$$28 - 25e^{-0.08 \times 0.3333} = \$3.66$$

Problem 9.3.

What is a lower bound for the price of a one-month European put option on a non-dividend-paying stock when the stock price is $12, the strike price is $15, and the risk-free interest rate is 6% per annum?

The lower bound is

$$15e^{-0.06 \times 0.08333} - 12 = \$2.93$$

Problem 9.4.

Give two reasons that the early exercise of an American call option on a non-dividend-paying stock is not optimal. The first reason should involve the time value of money. The second reason should apply even if interest rates are zero.

Delaying exercise delays the payment of the strike price. This means that the option holder is able to earn interest on the strike price for a longer period of time. Delaying exercise also provides insurance against the stock price falling below the strike price by the expiration date. Assume that the option holder has an amount of cash K and that interest rates are zero. Exercising early means that the option holder's position will be worth S_T at expiration. Delaying exercise means that it will be worth $\max(K, S_T)$ at expiration.

Problem 9.5.

"The early exercise of an American put is a trade-off between the time value of money and the insurance value of a put." Explain this statement.

An American put when held in conjunction with the underlying stock provides insurance. It guarantees that the stock can be sold for the strike price, K. If the put is exercised early, the insurance ceases. However, the option holder receives the strike price immediately and is able to earn interest on it between the time of the early exercise and the expiration date.

Problem 9.6.

Explain why an American call option on a dividend-paying stock is always worth at least as much as its intrinsic value. Is the same true of a European call option? Explain your answer.

An American call option can be exercised at any time. If it is exercised its holder gets the intrinsic value. It follows that an American call option must be worth at least its intrinsic value. A European call option can be worth less than its intrinsic value. Consider, for example, the situation where a stock is expected to provide a very high dividend during the life of an option. The price of the stock will decline as a result of the dividend. Because the European option can be exercised only after the dividend has been paid, its value may be less than the intrinsic value today.

Problem 9.7.

The price of a non-dividend paying stock is $19 and the price of a three-month European call option on the stock with a strike price of $20 is $1. The risk-free rate is 4% per annum. What is the price of a three-month European put option with a strike price of $20?

In this case $c = 1$, $T = 0.25$, $S_0 = 19$, $K = 20$, and $r = 0.04$. From put–call parity

$$p = c + Ke^{-rT} - S_0$$

or

$$p = 1 + 20e^{-0.04\times0.25} - 19 = 1.80$$

so that the European put price is $1.80.

Problem 9.8.

Explain why the arguments leading to put–call parity for European options cannot be used to give a similar result for American options.

When early exercise is not possible, we can argue that two portfolios that are worth the same at time T must be worth the same at earlier times. When early exercise is possible, the argument falls down. Suppose that $P + S > C + Ke^{-rT}$. This situation does not lead to an arbitrage opportunity. If we buy the call, short the put, and short the stock, we cannot be sure of the result because we do not know when the put will be exercised.

Problem 9.9.

What is a lower bound for the price of a six-month call option on a non-dividend-paying stock when the stock price is $80, the strike price is $75, and the risk-free interest rate is 10% per annum?

The lower bound is

$$80 - 75e^{-0.1\times0.5} = \$8.66$$

Problem 9.10

What is a lower bound for the price of a two-month European put option on a non-dividend-paying stock when the stock price is \$58, the strike price is \$65, and the risk-free interest rate is 5% per annum?

The lower bound is

$$65e^{-0.05\times 2/12} - 58 = \$6.46$$

Problem 9.11.

A four-month European call option on a dividend-paying stock is currently selling for \$5. The stock price is \$64, the strike price is \$60, and a dividend of \$0.80 is expected in one month. The risk-free interest rate is 12% per annum for all maturities. What opportunities are there for an arbitrageur?

The present value of the strike price is $60e^{-0.12\times 4/12} = \$57.65$. The present value of the dividend is $0.80e^{-0.12\times 1/12} = 0.79$. Because

$$5 < 64 - 57.65 - 0.79$$

the condition in equation (9.5) is violated. An arbitrageur should buy the option and short the stock. This generates $64 - 5 = \$59$. The arbitrageur invests \$0.79 of this at 12% for one month to pay the dividend of \$0.80 in one month. The remaining \$58.21 is invested for four months at 12%. Regardless of what happens a profit will materialize.

If the stock price declines below \$60 in four months, the arbitrageur loses the \$5 spent on the option but gains on the short position. The arbitrageur shorts when the stock price is \$64, has to pay dividends with a present value of \$0.79, and closes out the short position when the stock price is \$60 or less. Because \$57.65 is the present value of \$60, the short position generates at least $64 - 57.65 - 0.79 = \$5.56$ in present value terms. The present value of the arbitrageur's gain is therefore at least $5.56 - 5.00 = \$0.56$.

If the stock price is above \$60 at the expiration of the option, the option is exercised. The arbitrageur buys the stock for \$60 in four months and closes out the short position. The present value of the \$60 paid for the stock is \$57.65 and as before the dividend has a present value of \$0.79. The gain from the short position and the exercise of the option is therefore exactly $64 - 57.65 - 0.79 = \$5.56$. The arbitrageur's gain in present value terms is exactly $5.56 - 5.00 = \$0.56$.

Problem 9.12.

A one-month European put option on a non-dividend-paying stock is currently selling for \$2.50. The stock price is \$47, the strike price is \$50, and the risk-free interest rate is 6% per annum. What opportunities are there for an arbitrageur?

In this case the present value of the strike price is $50e^{-0.06\times 1/12} = 49.75$. Because

$$2.5 < 49.75 - 47.00$$

the condition in equation (9.2) is violated. An arbitrageur should borrow \$49.50 at 6% for one month, buy the stock, and buy the put option. This generates a profit in all circumstances.

If the stock price is above \$50 in one month, the option expires worthless, but the stock can be sold for at least \$50. A sum of \$50 received in one month has a present value of \$49.75 today. The strategy therefore generates profit with a present value of at least \$0.25.

If the stock price is below \$50 in one month the put option is exercised and the stock owned is sold for exactly \$50 (or \$49.75 in present value terms). The trading strategy therefore generates a profit of exactly \$0.25 in present value terms.

Problem 9.13.

Give an intuitive explanation of why the early exercise of an American put becomes more attractive as the risk-free rate increases and volatility decreases.

The early exercise of an American put is attractive when the interest earned on the strike price is greater than the insurance element lost. When interest rates increase, the value of the interest earned on the strike price increases making early exercise more attractive. When volatility decreases, the insurance element is less valuable. Again this makes early exercise more attractive.

Problem 9.14.

The price of a European call that expires in six months and has a strike price of \$30 is \$2. The underlying stock price is \$29, and a dividend of \$0.50 is expected in two months and again in five months. The term structure is flat, with all risk-free interest rates being 10%. What is the price of a European put option that expires in six months and has a strike price of \$30?

Using the notation in the chapter, put-call parity [equation (9.7)] gives

$$c + Ke^{-rT} + D = p + S_0$$

or

$$p = c + Ke^{-rT} + D - S_0$$

In this case

$$p = 2 + 30e^{-0.1\times 6/12} + (0.5e^{-0.1\times 2/12} + 0.5e^{-0.1\times 5/12}) - 29 = 2.51$$

In other words the put price is \$2.51.

Problem 9.15.

Explain carefully the arbitrage opportunities in Problem 9.14 if the European put price is \$3.

If the put price is \$3.00, it is too high relative to the call price. An arbitrageur should buy the call, short the put and short the stock. This generates $-2 + 3 + 29 = \$30$ in cash which is invested at 10%. Regardless of what happens a profit with a present value of $3.00 - 2.51 = \$0.49$ is locked in.

If the stock price is above \$30 in six months, the call option is exercised, and the put option expires worthless. The call option enables the stock to be bought for \$30, or $30e^{-0.10\times 6/12} =$ \$28.54 in present value terms. The dividends on the short position cost $0.5e^{-0.1\times 2/12} + 0.5e^{-0.1\times 5/12} =$ \$0.97 in present value terms so that there is a profit with a present value of $30 - 28.54 - 0.97 =$ \$0.49.

If the stock price is below \$30 in six months, the put option is exercised and the call option expires worthless. The short put option leads to the stock being bought for \$30, or $30e^{-0.10\times 6/12} =$ \$28.54 in present value terms. The dividends on the short position cost $0.5e^{-0.1\times 2/12} + 0.5e^{-0.1\times 5/12} =$ \$0.97 in present value terms so that there is a profit with a present value of $30 - 28.54 - 0.97 =$ \$0.49.

Problem 9.16.

The price of an American call on a non-dividend-paying stock is \$4. The stock price is \$31, the strike price is \$30, and the expiration date is in three months. The risk-free interest rate is 8%. Derive upper and lower bounds for the price of an American put on the same stock with the same strike price and expiration date.

From equation (9.4)

$$S_0 - K \le C - P \le S_0 - Ke^{-rT}$$

In this case

$$31 - 30 \le 4 - P \le 31 - 30e^{-0.08\times 0.25}$$

or

$$1.00 \le 4.00 - P \le 1.59$$

or

$$2.41 \le P \le 3.00$$

Upper and lower bounds for the price of an American put are therefore \$2.41 and \$3.00.

Problem 9.17.

Explain carefully the arbitrage opportunities in Problem 9.16 if the American put price is greater than the calculated upper bound.

If the American put price is greater than \$3.00 an arbitrageur can sell the American put, short the stock, and buy the American call. This realizes at least $3 + 31 - 4 =$ \$30 which can be invested at the risk-free interest rate. At some stage during the 3-month period either the American put or the American call will be exercised. The arbitrageur then pays \$30, receives the stock and closes out the short position. The cash flows to the arbitrageur are +\$30 at time zero and −\$30 at some future time. These cash flows have a positive present value.

Problem 9.18.

Prove the result in equation (9.4). (Hint: For the first part of the relationship consider (a) a portfolio consisting of a European call plus an amount of cash equal to K and (b) a portfolio consisting of an American put option plus one share.)

As in the text we use c and p to denote the European call and put option price, and C and P to denote the American call and put option prices. Because $P \geq p$, it follows from put–call parity that

$$P \geq c + Ke^{-rT} - S_0$$

and since $c = C$,

$$P \geq C + Ke^{-rT} - S_0$$

or

$$C - P \leq S_0 - Ke^{-rT}$$

For a further relationship between C and P, consider

Portfolio I: One European call option plus an amount of cash equal to K.

Portfolio J: One American put option plus one share.

Both options have the same exercise price and expiration date. Assume that the cash in portfolio I is invested at the risk-free interest rate. If the put option is not exercised early portfolio J is worth

$$\max(S_T,\ K)$$

at time T. Portfolio I is worth

$$\max(S_T - K,\ 0) + Ke^{rT} = \max(S_T, K) - K + Ke^{rT}$$

at this time. Portfolio I is therefore worth more than portfolio J. Suppose next that the put option in portfolio J is exercised early, say, at time τ. This means that portfolio J is worth K at time τ. However, even if the call option were worthless, portfolio I would be worth $Ke^{r\tau}$ at time τ. It follows that portfolio I is worth at least as much as portfolio J in all circumstances. Hence

$$c + K \geq P + S_0$$

Since $c = C$,

$$C + K \geq P + S_0$$

or

$$C - P \geq S_0 - K$$

Combining this with the other inequality derived above for $C - P$, we obtain

$$S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$$

Problem 9.19.

Prove the result in equation (9.8). (Hint: For the first part of the relationship consider (a) a portfolio consisting of a European call plus an amount of cash equal to $D + K$ and (b) a portfolio consisting of an American put option plus one share.)

As in the text we use c and p to denote the European call and put option price, and C and P to denote the American call and put option prices. The present value of the dividends will be denoted by D. As shown in the answer to Problem 9.18, when there are no dividends

$$C - P \leq S_0 - Ke^{-rT}$$

Dividends reduce C and increase P. Hence this relationship must also be true when there are dividends.

For a further relationship between C and P, consider

Portfolio I: one European call option plus an amount of cash equal to $D + K$

Portfolio J: one American put option plus one share

Both options have the same exercise price and expiration date. Assume that the cash in portfolio I is invested at the risk-free interest rate. If the put option is not exercised early, portfolio J is worth

$$\max(S_T,\ K) + De^{rT}$$

at time T. Portfolio I is worth

$$\max(S_T - K,\ 0) + (D + K)e^{rT} = \max(S_T,\ K) + De^{rT} + Ke^{rT} - K$$

at this time. Portfolio I is therefore worth more than portfolio J. Suppose next that the put option in portfolio J is exercised early, say, at time τ. This means that portfolio J is worth at most $K + De^{r\tau}$ at time τ. However, even if the call option were worthless, portfolio I would be worth $(D + K)e^{r\tau}$ at time τ. It follows that portfolio I is worth more than portfolio J in all circumstances. Hence

$$c + D + K \geq P + S_0$$

Because $C \geq c$

$$C - P \geq S_0 - D - K$$

Problem 9.20.

Consider a 5-year employee stock option on a non-dividend-paying stock. The option can be exercised at any time after the end of the first year. Unlike a regular exchange-traded call option, the employee stock option cannot be sold. What is the likely impact of this restriction on the early exercise decision?

Executive stock options may be exercised early because the executive needs the cash or because he or she is uncertain about the company's future prospects. Regular call options can be sold in the market in either of these two situations, but executive stock options cannot be sold. In theory an executive can short the company's stock as an alternative to exercising. In practice this is not usually encouraged and may even be illegal.

Problem 9.21.

Use the software DerivaGem to verify that Figures 9.1 and 9.2 are correct.

The graphs can be produced from the first worksheet in DerivaGem. Select Equity as the Underlying Type. Select Analytic European as the Option Type. Input stock price as 50, volatility as 30%, risk-free rate as 5%, time to exercise as 1 year, and exercise price as 50. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Do not select the implied volatility button. Hit the *Enter* key and click on calculate. DerivaGem will show the price of the option as 7.15562248. Move to the Graph Results on the right hand side of the worksheet. Enter Option Price for the vertical axis and Asset price for the horizontal axis. Choose the minimum strike price value as 10 (software will not accept 0) and the maximum strike price value as 100. Hit *Enter* and click on *Draw Graph*. This will produce Figure 9.1a. Figures 9.1c, 9.1e, 9.2a, and 9.2c can be produced similarly by changing the horizontal axis. By selecting put instead of call and recalculating the rest of the figures can be produced. You are encouraged to experiment with this worksheet. Try different parameter values and different types of options.

ASSIGNMENT QUESTIONS

Problem 9.22.

A European call option and put option on a stock both have a strike price of $20 and an expiration date in three months. Both sell for $3. The risk-free interest rate is 10% per annum, the current stock price is $19, and a $1 dividend is expected in one month. Identify the arbitrage opportunity open to a trader.

If the call is worth $3, put-call parity shows that the put should be worth

$$3 + 20e^{-0.10\times 3/12} + e^{-0.1\times 1/12} - 19 = 4.50$$

This is greater than $3. The put is therefore undervalued relative to the call. The correct arbitrage strategy is to buy the put, buy the stock, and short the call. This costs $19. If the stock price in three months is greater than $20, the call is exercised. If it is less than $20, the put is exercised. In either case the arbitrageur sells the stock for $20 and collects the $1 dividend in one month. The present value of the gain to the arbitrageur is

$$-3 - 19 + 3 + 20e^{-0.10\times 3/12} + e^{-0.1\times 1/12} = 1.50$$

Problem 9.23.

Suppose that c_1, c_2, and c_3 are the prices of European call options with strike prices K_1, K_2, and K_3, respectively, where $K_3 > K_2 > K_1$ and $K_3 - K_2 = K_2 - K_1$. All options have the same maturity. Show that

$$c_2 \leq 0.5(c_1 + c_3)$$

(Hint: Consider a portfolio that is long one option with strike price K_1, long one option with strike price K_3, and short two options with strike price K_2.)

Consider a portfolio that is long one option with strike price K_1, long one option with strike price K_3, and short two options with strike price K_2. The value of the portfolio can be worked out in four different situations

$$\begin{aligned}
S_T \le K_1 :&\ \text{Portfolio Value } = 0\\
K_1 < S_T \le K_2 :&\ \text{Portfolio Value } = S_T - K_1\\
K_2 < S_T \le K_3 :&\ \text{Portfolio Value } = S_T - K_1 - 2(S_T - K_2)\\
&\qquad\qquad\qquad\quad = K_2 - K_1 - (S_T - K_2) \ge 0\\
S_T > K_3 :&\ \text{Portfolio Value } = S_T - K_1 - 2(S_T - K_2) + S_T - K_3\\
&\qquad\qquad\qquad\quad = K_2 - K_1 - (K_3 - K_2)\\
&\qquad\qquad\qquad\quad = 0
\end{aligned}$$

The value is always either positive or zero at the expiration of the option. In the absence of arbitrage possibilities it must be positive or zero today. This means that

$$c_1 + c_3 - 2c_2 \ge 0$$

or

$$c_2 \le 0.5(c_1 + c_3)$$

Note that students often think they have proved this by writing down

$$c_1 \le S_0 - K_1 e^{-rT}$$

$$2c_2 \le 2(S_0 - K_2 e^{-rT})$$

$$c_3 \le S_0 - K_3 e^{-rT}$$

and subtracting the middle inequality from the sum of the other two. But they are deceiving themselves. Inequality relationships cannot be subtracted. For example, $9 > 8$ and $5 > 2$, but it is not true that $9 - 5 > 8 - 2$

Problem 9.24.

What is the result corresponding to that in Problem 9.23 for European put options?

The corresponding result is

$$p_2 \le 0.5(p_1 + p_3)$$

where p_1, p_2 and p_3 are the prices of European put option with the same maturities and strike prices K_1, K_2 and K_3 respectively. This can be proved from the result in Problem 9.23 using put-call parity. Alternatively we can consider a portfolio consisting of a long position in a put option with strike price K_1, a long position in a put option with strike price K_3, and a short position in two put options with strike price K_2. The value of this portfolio in different situations is given as follows

$$S_T \le K_1 : \text{ Portfolio Value } = K_1 - S_T - 2(K_2 - S_T) + K_3 - S_T$$

$$
\begin{aligned}
& & &= K_3 - K_2 - (K_2 - K_1) \\
& & &= 0 \\
K_1 < S_T \le K_2 :& \quad \text{Portfolio Value} & &= K_3 - S_T - 2(K_2 - S_T) \\
& & &= K_3 - K_2 - (K_2 - S_T) \\
& & &\ge 0 \\
K_2 < S_T \le K_3 :& \quad \text{Portfolio Value} & &= K_3 - S_T \\
S_T > K_3 :& \quad \text{Portfolio Value} & &= 0
\end{aligned}
$$

Because the portfolio value is always zero or positive at some future time the same must be true today. Hence

$$p_1 + p_3 - 2p_2 \ge 0$$

or

$$p_2 \le 0.5(p_1 + p_3)$$

Problem 9.25.

Suppose that you are the manager and sole owner of a highly leveraged company. All the debt will mature in one year. If at that time the value of the company is greater than the face value of the debt, you will pay off the debt. If the value of the company is less than the face value of the debt, you will declare bankruptcy and the debt holders will own the company.

a. *Express your position as an option on the value of the company.*
b. *Express the position of the debt holders in terms of options on the value of the company.*
c. *What can you do to increase the value of your position?*

(a) Suppose V is the value of the company and D is the face value of the debt. The value of the manager's position in one year is

$$\max(V - D,\ 0)$$

This is the payoff from a call option on V with strike price D.

(b) The debt holders get

$$\min(V,\ D)$$

$$= D - \max(D - V,\ 0)$$

Since $\max(D - V,\ 0)$ is the payoff from a put option on V with strike price D, the debt holders have in effect made a risk-free loan (worth D at maturity with certainty) and written a put option on the value of the company with strike price D. The position of the debt holders in one year can also be characterized as

$$V - \max(V - D,\ 0)$$

This is a long position in the assets of the company combined with a short position in a call option on the assets with a strike price of D. The equivalence of the two characterizations can be presented as an application of put–call parity. (See Business Snapshot 9.1.)

(c) The manager can increase the value of his or her position by increasing the value of the call option in (a). It follows that the manager should attempt to increase both V and the volatility of V. To see why increasing the volatility of V is beneficial, imagine what happens when there are large changes in V. If V increases, the manager benefits to the full extent of the change. If V decreases, much of the downside is absorbed by the company's lenders.

Problem 9.26.

Consider an option on a stock when the stock price is \$41, the strike price is \$40, the risk-free rate is 6%, the volatility is 35%, and the time to maturity is 1 year. Assume that a dividend of \$0.50 is expected after six months.

a. *Use DerivaGem to value the option assuming it is a European call.*
b. *Use DerivaGem to value the option assuming it is a European put.*
c. *Verify that put–call parity holds.*
d. *Explore using DerivaGem what happens to the price of the options as the time to maturity becomes very large. For this purpose assume there are no dividends. Explain the results you get.*

DerivaGem shows that the price of the call option is 6.9686 and the price of the put option is 4.1244. In this case

$$c + D + Ke^{-rT} = 6.9686 + 0.5e^{-0.06\times 0.5} + 40e^{-0.06\times 1} = 45.1244$$

Also

$$p + S = 4.1244 + 41 = 45.1244$$

As the time to maturity becomes very large and there are no dividends, the price of the call option approaches the stock price of 41. (For example when $T = 100$ it is 40.94.) This is because the call option can be regarded as a position in the stock where the price does not have to be paid for a very long time. The present value of what has to be paid is close to zero. As the time to maturity becomes very large the price of the European put option becomes close to zero. (For example when $T = 100$ it is 0.04.) This is because the present value of what might be received from the put option becomes close to zero.

CHAPTER 10

Trading Strategies Involving Options

Notes for the Instructor

This chapter covers various ways in which traders can form portfolios of calls and puts to get interesting payoff patterns. For ease of exposition, the time value of money is ignored in payoff diagrams and payoff tables.

Students usually enjoy the chapter. As each spread strategy is covered, I like to use put–call parity to relate the cost of the spread created using calls to the cost of a spread created using puts. (See Problems 10.8 and 10.11) This reinforces the Chapter 9 material on put–call parity. It can be useful to cover Business Snapshot 10.1 in class.

Problem 10.19 can be used for class discussion. Problems 10.20, 10.21, 10.22, and 10.23 can be used as hand-in assignments.

QUESTIONS AND PROBLEMS

Problem 10.1.

What is meant by a protective put? What position in call options is equivalent to a protective put?

A protective put consists of a long position in a put option combined with a long position in the underlying shares. It is equivalent to a long position in a call option plus a certain amount of cash. This follows from put–call parity:

$$p + S_0 = c + Ke^{-rT} + D$$

Problem 10.2.

Explain two ways in which a bear spread can be created.

A bear spread can be created using two call options with the same maturity and different strike prices. The investor shorts the call option with the lower strike price and buys the call option with the higher strike price. A bear spread can also be created using two put options with the same maturity and different strike prices. In this case, the investor shorts the put option with the lower strike price and buys the put option with the higher strike price.

Problem 10.3.

When is it appropriate for an investor to purchase a butterfly spread?

A butterfly spread involves a position in options with three different strike prices (K_1, K_2, and K_3). A butterfly spread should be purchased when the investor considers that the price of the underlying stock is likely to stay close to the central strike price, K_2.

Problem 10.4.

Call options on a stock are available with strike prices of \$15, $\$17\frac{1}{2}$, and \$20 and expiration dates in three months. Their prices are \$4, \$2, and $\$\frac{1}{2}$, respectively. Explain how the options can be used to create a butterfly spread. Construct a table showing how profit varies with stock price for the butterfly spread.

An investor can create a butterfly spread by buying call options with strike prices of \$15 and \$20 and selling two call options with strike prices of $\$17\frac{1}{2}$. The initial investment is $4 + \frac{1}{2} - 2 \times 2 = \$\frac{1}{2}$. The following table shows the variation of profit with the final stock price:

Stock Price S_T	Profit
$S_T < 15$	$-\frac{1}{2}$
$15 < S_T < 17\frac{1}{2}$	$(S_T - 15) - \frac{1}{2}$
$17\frac{1}{2} < S_T < 20$	$(20 - S_T) - \frac{1}{2}$
$S_T > 20$	$-\frac{1}{2}$

Problem 10.5.

What trading strategy creates a reverse calendar spread?

A reverse calendar spread is created by buying a short-maturity option and selling a long-maturity option, both with the same strike price.

Problem 10.6.

What is the difference between a strangle and a straddle?

Both a straddle and a strangle are created by combining a long position in a call with a long position in a put. In a straddle the two have the same strike price and expiration date. In a strangle they have different strike prices and the same expiration date.

Problem 10.7.

A call option with a strike price of \$50 costs \$2. A put option with a strike price of \$45 costs \$3. Explain how a strangle can be created from these two options. What is the pattern of profits from the strangle?

A strangle is created by buying both options. The pattern of profits is as follows:

Stock Price S_T	Profit
$S_T < 45$	$(45 - S_T) - 5$
$45 < S_T < 50$	-5
$S_T > 50$	$(S_T - 50) - 5$

Problem 10.8.

Use put–call parity to relate the initial investment for a bull spread created using calls to the initial investment for a bull spread created using puts.

A bull spread using calls provides a profit pattern with the same general shape as a bull spread using puts (see Figures 10.2 and 10.3 in the text). Define p_1 and c_1 as the prices of put and call with strike price K_1 and p_2 and c_2 as the prices of a put and call with strike price K_2. From put-call parity

$$p_1 + S = c_1 + K_1 e^{-rT}$$

$$p_2 + S = c_2 + K_2 e^{-rT}$$

Hence:

$$p_1 - p_2 = c_1 - c_2 - (K_2 - K_1)e^{-rT}$$

This shows that the initial investment when the spread is created from puts is less than the initial investment when it is created from calls by an amount $(K_2 - K_1)e^{-rT}$. In fact as mentioned in the text the initial investment when the bull spread is created from puts is negative, while the initial investment when it is created from calls is positive.

The profit when calls are used to create the bull spread is higher than when puts are used by $(K_2 - K_1)(1 - e^{-rT})$. This reflects the fact that the call strategy involves an additional risk-free investment of $(K_2 - K_1)e^{-rT}$ over the put strategy. This earns interest of $(K_2 - K_1)e^{-rT}(e^{rT} - 1) = (K_2 - K_1)(1 - e^{-rT})$.

Problem 10.9.

Explain how an aggressive bear spread can be created using put options.

An aggressive bull spread using call options is discussed in the text. Both of the options used have relatively high strike prices. Similarly, an aggressive bear spread can be created using put options. Both of the options should be out of the money (that is, they should have relatively low strike prices). The spread then costs very little to set up because both of the puts are worth close to zero. In most circumstances the spread will provide zero payoff. However, there is a small chance that the stock price will fall fast so that on expiration both options will be in the money. The spread then provides a payoff equal to the difference between the two strike prices, $K_2 - K_1$.

Problem 10.10.

Suppose that put options on a stock with strike prices $30 and $35 cost $4 and $7, respectively. How can the options be used to create (a) a bull spread and (b) a bear spread? Construct a table that shows the profit and payoff for both spreads.

A bull spread is created by buying the $30 put and selling the $35 put. This strategy gives rise to an initial cash inflow of $3. The outcome is as follows:

Stock Price	Payoff	Profit
$S_T \geq 35$	0	3
$30 \leq S_T < 35$	$S_T - 35$	$S_T - 32$
$S_T < 30$	-5	-2

A bear spread is created by selling the $30 put and buying the $35 put. This strategy costs $3 initially. The outcome is as follows:

Stock Price	Payoff	Profit
$S_T \geq 35$	0	-3
$30 \leq S_T < 35$	$35 - S_T$	$32 - S_T$
$S_T < 30$	5	2

Problem 10.11.

Use put–call parity to show that the cost of a butterfly spread created from European puts is identical to the cost of a butterfly spread created from European calls.

Define c_1, c_2, and c_3 as the prices of calls with strike prices K_1, K_2 and K_3. Define p_1, p_2 and p_3 as the prices of puts with strike prices K_1, K_2 and K_3. With the usual notation

$$c_1 + K_1 e^{-rT} = p_1 + S$$

$$c_2 + K_2 e^{-rT} = p_2 + S$$

$$c_3 + K_3 e^{-rT} = p_3 + S$$

Hence

$$c_1 + c_3 - 2c_2 + (K_1 + K_3 - 2K_2)e^{-rT} = p_1 + p_3 - 2p_2$$

Because $K_2 - K_1 = K_3 - K_2$, it follows that $K_1 + K_3 - 2K_2 = 0$ and

$$c_1 + c_3 - 2c_2 = p_1 + p_3 - 2p_2$$

The cost of a butterfly spread created using European calls is therefore exactly the same as the cost of a butterfly spread created using European puts.

Problem 10.12.

A call with a strike price of \$60 costs \$6. A put with the same strike price and expiration date costs \$4. Construct a table that shows the profit from a straddle. For what range of stock prices would the straddle lead to a loss?

A straddle is created by buying both the call and the put. This strategy costs \$10. The profit/loss is shown in the following table:

Stock Price	Payoff	Profit
$S_T > 60$	$S_T - 60$	$S_T - 70$
$S_T \leq 60$	$60 - S_T$	$50 - S_T$

This shows that the straddle will lead to a loss if the final stock price is between \$50 and \$70.

Problem 10.13.

Construct a table showing the payoff from a bull spread when puts with strike prices K_1 and K_2 are used ($K_2 > K_1$).

The bull spread is created by buying a put with strike price K_1 and selling a put with strike price K_2. The payoff is calculated as follows:

Stock Price Range	Payoff from Long Put Option	Payoff from Short Put Option	Total Payoff
$S_T \geq K_2$	0	0	0
$K_1 < S_T < K_2$	0	$S_T - K_2$	$-(K_2 - S_T)$
$S_T \leq K_1$	$K_1 - S_T$	$S_T - K_2$	$-(K_2 - K_1)$

Problem 10.14.

An investor believes that there will be a big jump in a stock price, but is uncertain as to the direction. Identify six different strategies the investor can follow and explain the differences among them.

Possible strategies are:

Strangle
Straddle
Strip
Strap
Reverse calendar spread
Reverse butterfly spread

The strategies all provide positive profits when there are large stock price moves. A strangle is less expensive than a straddle, but requires a bigger move in the stock price in order to provide a positive profit. Strips and straps are more expensive than straddles but provide bigger profits in certain circumstances. A strip will provide a bigger profit when there is a large downward stock price move. A strap will provide a bigger profit when there is a large upward stock price move. In the case of strangles, straddles, strips and straps, the profit increases as the size of the stock price movement increases. By contrast in a reverse calendar spread and a reverse butterfly spread there is a maximum potential profit regardless of the size of the stock price movement.

Problem 10.15.

How can a forward contract on a stock with a particular delivery price and delivery date be created from options?

Suppose that the delivery price is K and the delivery date is T. The forward contract is created by buying a European call and selling a European put when both options have strike price K and exercise date T. This portfolio provides a payoff of $S_T - K$ under all circumstances where S_T is the stock price at time T. Suppose that F_0 is the forward price. If $K = F_0$, the forward contract that is created has zero value. This shows that the price of a call equals the price of a put when the strike price is F_0.

Problem 10.16.

"A box spread comprises four options. Two can be combined to create a long forward position and two can be combined to create a short forward position." Explain this statement.

A box spread is a bull spread created using calls and a bear spread created using puts. With the notation in the text it consists of a) a long call with strike K_1, b) a short call with strike K_2, c) a long put with strike K_2, and d) a short put with strike K_1. a) and d) give a long forward contract with delivery price K_1; b) and c) give a short forward contract with delivery price K_2. The two forward contracts taken together give the payoff of $K_2 - K_1$.

Problem 10.17.

What is the result if the strike price of the put is higher than the strike price of the call in a strangle?

The result is shown in Figure S10.1. The profit pattern from a long position in a call and a put when the put has a higher strike price than a call is much the same as when the call has a higher strike price than the put. Both the initial investment and the final payoff are much higher in the first case.

Problem 10.18.

One Australian dollar is currently worth \$0.64. A one-year butterfly spread is set up using European call options with strike prices of \$0.60, \$0.65, and \$0.70. The risk-free

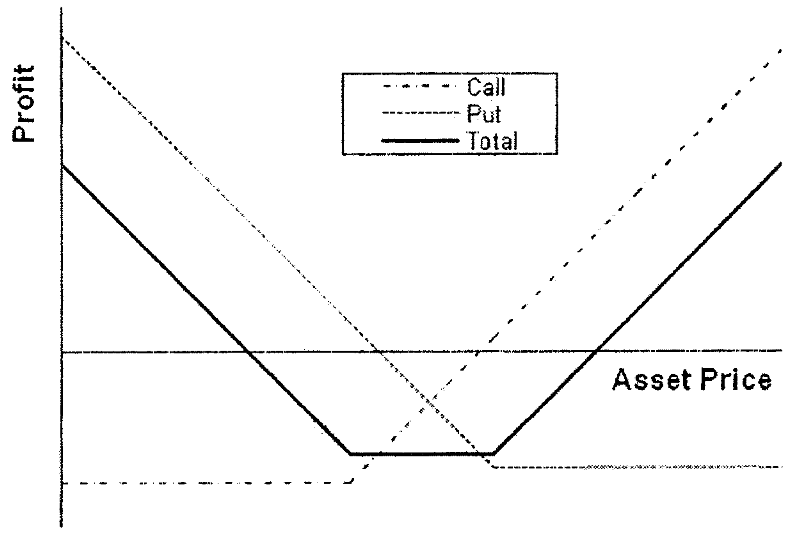


Figure S10.1 Profit Pattern in Problem 10.17

interest rates in the United States and Australia are 5% and 4% respectively, and the volatility of the exchange rate is 15%. Use the DerivaGem software to calculate the cost of setting up the butterfly spread position. Show that the cost is the same if European put options are used instead of European call options.

To use DerivaGem select the first worksheet and choose Currency as the Underlying Type. Select Analytic European as the Option Type. Input exchange rate as 0.64, volatility as 15%, risk-free rate as 5%, foreign risk-free interest rate as 4%, time to exercise as 1 year, and exercise price as 0.60. Select the button corresponding to call. Do not select the implied volatility button. Hit the *Enter* key and click on calculate. DerivaGem will show the price of the option as 0.0618. Change the exercise price to 0.65, hit *Enter*, and click on calculate again. DerivaGem will show the value of the option as 0.0352. Change the exercise price to 0.70, hit *Enter*, and click on calculate. DerivaGem will show the value of the option as 0.0181.

Now select the button corresponding to put and repeat the procedure. DerivaGem shows the values of puts with strike prices 0.60, 0.65, and 0.70 to be 0.0176, 0.0386, and 0.0690, respectively.

The cost of setting up the butterfly spread when calls are used is therefore

$$0.0618 + 0.0181 - 2 \times 0.0352 = 0.0095$$

The cost of setting up the butterfly spread when puts are used is

$$0.0176 + 0.0690 - 2 \times 0.0386 = 0.0094$$

Allowing for rounding errors these two are the same.

ASSIGNMENT QUESTIONS

Problem 10.19.

Three put options on a stock have the same expiration date and strike prices of $55, $60, and $65. The market prices are $3, $5, and $8, respectively. Explain how a butterfly spread can be created. Construct a table showing the profit from the strategy. For what range of stock prices would the butterfly spread lead to a loss?

A butterfly spread is created by buying the $55 put, buying the $65 put and selling two of the $60 puts. This costs $3 + 8 - 2 \times 5 =$ $1 initially. The following table shows the profit/loss from the strategy.

Stock Price	Payoff	Profit
$S_T \geq 65$	0	−1
$60 \leq S_T < 65$	$65 - S_T$	$64 - S_T$
$55 \leq S_T < 60$	$S_T - 55$	$S_T - 56$
$S_T < 55$	0	−1

The butterfly spread leads to a loss when the final stock price is greater then $64 or less than $56.

Problem 10.20.

A diagonal spread is created by buying a call with strike price K_2 and exercise date T_2 and selling a call with strike price K_1 and exercise date T_1 ($T_2 > T_1$). Draw a diagram showing the profit when (a) $K_2 > K_1$ and (b) $K_2 < K_1$.

There are two alternative profit patterns for part (a). These are shown in Figures M10.1 and M10.2. In Figure M10.1 the long maturity (high strike price) option is worth more than the short maturity (low strike price) option. In Figure M10.2 the reverse is true. There is no ambiguity about the profit pattern for part (b). This is shown in Figure M10.3.

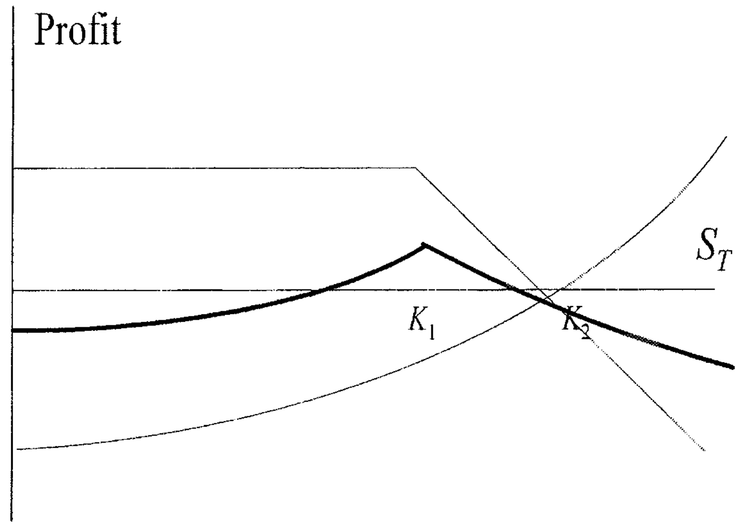


Figure M10.1 Investor's Profit/Loss in Problem 10.20a when long maturity call is worth more than short maturity call

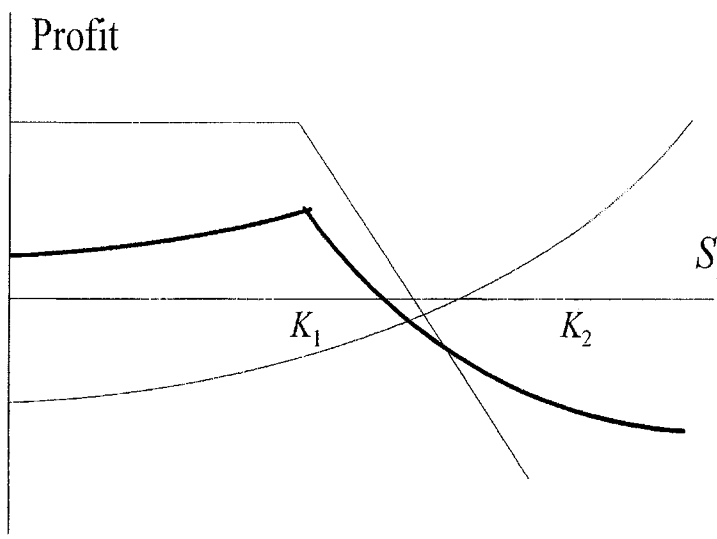


Figure M10.2 Investor's Profit/Loss in Problem 10.20a when short maturity call is worth more than long maturity call

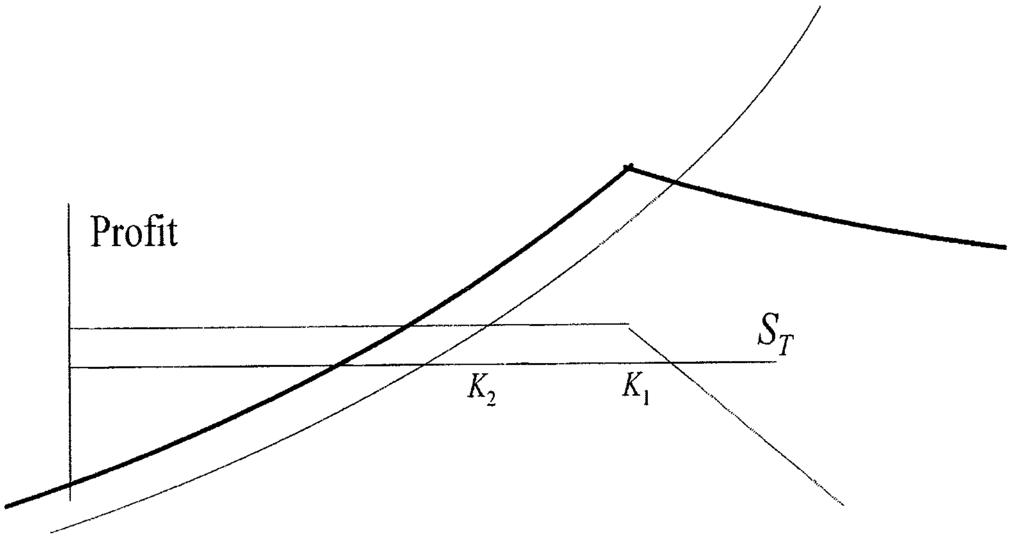


Figure M10.3 Investor's Profit/Loss in Problem 10.20b

Problem 10.21.

Draw a diagram showing the variation of an investor's profit and loss with the terminal stock price for a portfolio consisting of

a. *One share and a short position in one call option*
b. *Two shares and a short position in one call option*
c. *One share and a short position in two call options*
d. *One share and a short position in four call options*

In each case, assume that the call option has an exercise price equal to the current stock price.

The variation of an investor's profit/loss with the terminal stock price for each of the four strategies is shown in Figure M10.4. In each case the dotted line shows the profits from the components of the investor's position and the solid line shows the total net profit.

Problem 10.22.

Suppose that the price of a non-dividend-paying stock is $32, its volatility is 30%, and the risk-free rate for all maturities is 5% per annum. Use DerivaGem to calculate the cost of setting up the following positions. In each case provide a table showing the relationship between profit and final stock price. Ignore the impact of discounting.

a. *A bull spread using European call options with strike prices of $25 and $30 and a maturity of six months.*
b. *A bear spread using European put options with strike prices of $25 and $30 and a maturity of six months*
c. *A butterfly spread using European call options with strike prices of $25, $30, and $35 and a maturity of one year.*
d. *A butterfly spread using European put options with strike prices of $25, $30, and $35 and a maturity of one year.*
e. *A straddle using options with a strike price of $30 and a six-month maturity.*

f. A strangle using options with strike prices of \$25 and \$35 and a six-month maturity.

(a) A call option with a strike price of 25 costs 7.90 and a call option with a strike price of 30 costs 4.18. The cost of the bull spread is therefore $7.90 - 4.18 = 3.72$. The profits ignoring the impact of discounting are

Stock Price Range	Profit
$S_T \leq 25$	-3.72
$25 < S_T < 30$	$S_T - 28.72$
$S_T \geq 30$	1.28

(b) A put option with a strike price of 25 costs 0.28 and a put option with a strike price of 30 costs 1.44. The cost of the bear spread is therefore $1.44 - 0.28 = 1.16$. The profits ignoring the impact of discounting are

Stock Price Range	Profit
$S_T \leq 25$	$+3.84$
$25 < S_T < 30$	$28.84 - S_T$
$S_T \geq 30$	-1.16

(c) Call options with maturities of one year and strike prices of 25, 30, and 35 cost 8.92, 5.60, and 3.28, respectively. The cost of the butterfly spread is therefore $8.92 + 3.28 - 2 \times 5.60 = 1.00$. The profits ignoring the impact of discounting are

Stock Price Range	Profit
$S_T \leq 25$	-1.00
$25 < S_T < 30$	$S_T - 26.00$
$30 \leq S_T < 35$	$34.00 - S_T$
$S_T \geq 35$	-1.00

(d) Put options with maturities of one year and strike prices of 25, 30, and 35 cost 0.70, 2.14, 4.57, respectively. The cost of the butterfly spread is therefore $0.70 + 4.57 - 2 \times 2.14 = 0.99$. Allowing for rounding errors, this is the same as in (c). The profits are the same as in (c).

(e) A call option with a strike price of 30 costs 4.18. A put option with a strike price of 30 costs 1.44. The cost of the straddle is therefore $4.18 + 1.44 = 5.62$. The profits ignoring the impact of discounting are

Stock Price Range	Profit
$S_T \leq 30$	$24.58 - S_T$
$S_T > 30$	$S_T - 35.62$

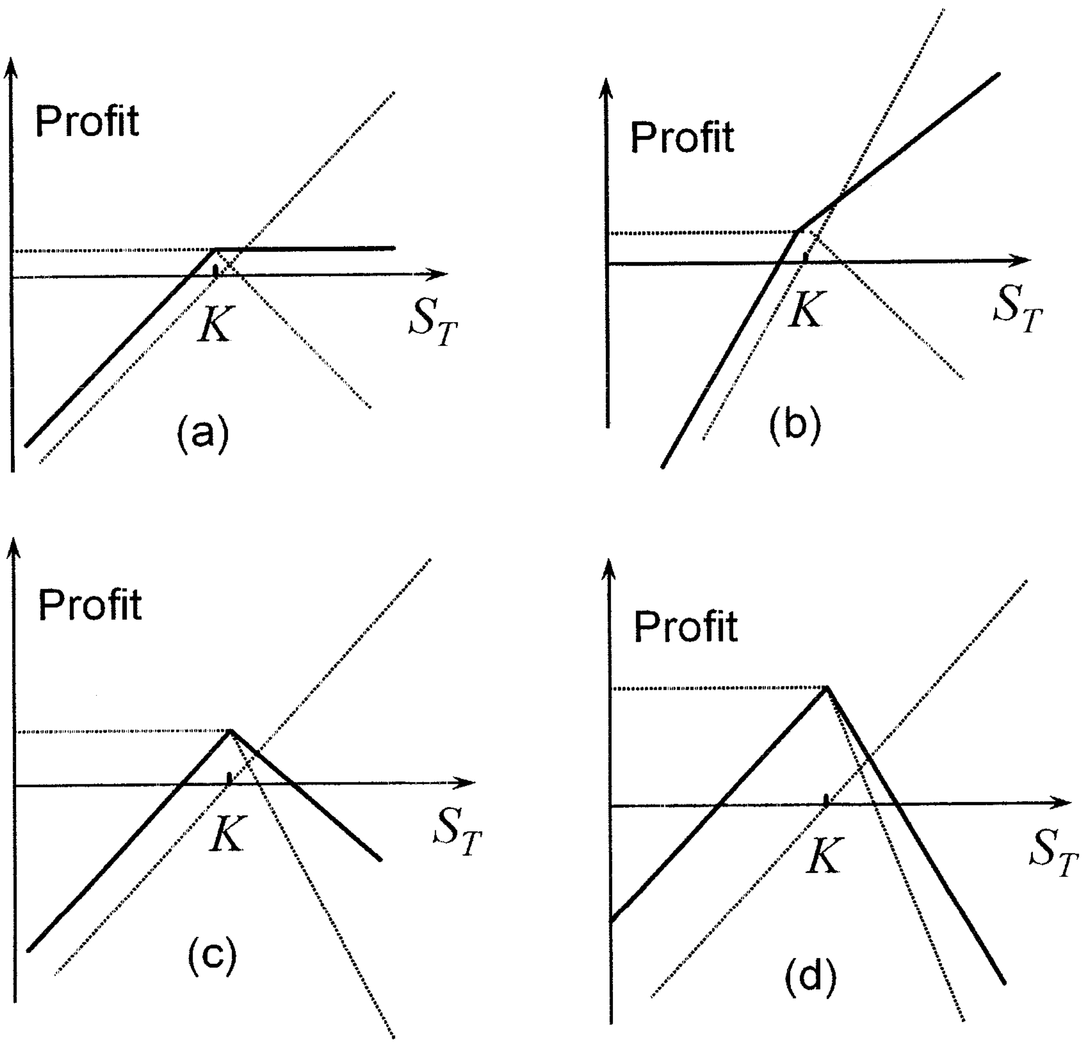


Figure M10.4 Investor's Profit/Loss in Problem 10.21

(f) A six-month call option with a strike price of 35 costs 1.85. A six-month put option with a strike price of 25 costs 0.28. The cost of the strangle is therefore $1.85 + 0.28 = 2.13$. The profits ignoring the impact of discounting are

Stock Price Range	Profit
$S_T \le 25$	$22.87 - S_T$
$25 < S_T < 35$	-2.13
$S_T \ge 35$	$S_T - 37.13$

Problem 10.23.

What trading position is created from a long strangle and a short straddle when both have the same time to maturity? Assume that the strike price in the straddle is half way between the two strike prices in the strangle.

A butterfly spread is created.

CHAPTER 11
Binomial Trees

Notes for the Instructor

This chapter discusses binomial trees. It enables some of the key concepts in option valuation to be introduced at a relatively early stage in a course. It includes material on the use of binomial trees for index options, currency options, and futures options (see Section 11.9).

The one-step binomial model can be used to demonstrate both no-arbitrage and risk-neutral valuation arguments. I like to first go through the arguments using the numerical example in the text and then generalize them by introducing some notation. After two- and three-step trees have been covered students should have a good appreciation of the way in which multistep trees are used to value options. DerivaGem provides a convenient way of displaying trees in class. The material on delta serves as an introduction to the hedging material in Chapter 17.

Any of Problems 11.16 to 11.22 can be used as assignments. I usually discuss 11.22 in class.

QUESTIONS AND PROBLEMS

Problem 11.1.

A stock price is currently \$40. It is known that at the end of one month it will be either \$42 or \$38. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-month European call option with a strike price of \$39?

Consider a portfolio consisting of

-1 :	Call option
$+\Delta$:	Shares

If the stock price rises to \$42, the portfolio is worth $42\Delta - 3$. If the stock price falls to \$38, it is worth 38Δ. These are the same when

$$42\Delta - 3 = 38\Delta$$

or $\Delta = 0.75$. The value of the portfolio in one month is 28.5 for both stock prices. Its value today must be the present value of 28.5, or $28.5e^{-0.08\times 0.08333} = 28.31$. This means that

$$-f + 40\Delta = 28.31$$

where f is the call price. Because $\Delta = 0.75$, the call price is $40 \times 0.75 - 28.31 = \$1.69$. As an alternative approach, we can calculate the probability, p, of an up movement in a risk-neutral world. This must satisfy:

$$42p + 38(1-p) = 40e^{0.08\times 0.08333}$$

so that

$$4p = 40e^{0.08 \times 0.08333} - 38$$

or $p = 0.5669$. The value of the option is then its expected payoff discounted at the risk-free rate:

$$[3 \times 0.5669 + 0 \times 0.4331]e^{-0.08 \times 0.08333} = 1.69$$

or \$1.69. This agrees with the previous calculation.

Problem 11.2.

Explain the no-arbitrage and risk-neutral valuation approaches to valuing a European option using a one-step binomial tree.

In the no-arbitrage approach, we set up a riskless portfolio consisting of a position in the option and a position in the stock. By setting the return on the portfolio equal to the risk-free interest rate, we are able to value the option. When we use risk-neutral valuation, we first choose probabilities for the branches of the tree so that the expected return on the stock equals the risk-free interest rate. We then value the option by calculating its expected payoff and discounting this expected payoff at the risk-free interest rate.

Problem 11.3.

What is meant by the delta of a stock option?

The delta of a stock option measures the sensitivity of the option price to the price of the stock when small changes are considered. Specifically, it is the ratio of the change in the price of the stock option to the change in the price of the underlying stock.

Problem 11.4.

A stock price is currently \$50. It is known that at the end of six months it will be either \$45 or \$55. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a six-month European put option with a strike price of \$50?

Consider a portfolio consisting of

-1 :	Put option
$+\Delta$:	Shares

If the stock price rises to \$55, this is worth 55Δ. If the stock price falls to \$45, the portfolio is worth $45\Delta - 5$. These are the same when

$$45\Delta - 5 = 55\Delta$$

or $\Delta = -0.50$. The value of the portfolio in six months is -27.5 for both stock prices. Its value today must be the present value of -27.5, or $-27.5e^{-0.1 \times 0.5} = -26.16$. This means that

$$-f + 50\Delta = -26.16$$

where f is the put price. Because $\Delta = -0.50$, the put price is \$1.16. As an alternative approach we can calculate the probability, p, of an up movement in a risk-neutral world. This must satisfy:

$$55p + 45(1 - p) = 50e^{0.1 \times 0.5}$$

so that

$$10p = 50e^{0.1\times0.5} - 45$$

or $p = 0.7564$. The value of the option is then its expected payoff discounted at the risk-free rate:

$$[0 \times 0.7564 + 5 \times 0.2436]e^{-0.1\times0.5} = 1.16$$

or \$1.16. This agrees with the previous calculation.

Problem 11.5.

A stock price is currently \$100. Over each of the next two six-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-year European call option with a strike price of \$100?

In this case $u = 1.10$, $d = 0.90$, $\Delta t = 0.5$, and $r = 0.08$, so that

$$p = \frac{e^{0.08\times0.5} - 0.90}{1.10 - 0.90} = 0.7041$$

The tree for stock price movements is shown in Figure S11.1. We can work back from the end of the tree to the beginning, as indicated in the diagram, to give the value of the option as \$9.61. The option value can also be calculated directly from equation (11.10):

$$[0.7041^2 \times 21 + 2 \times 0.7041 \times 0.2959 \times 0 + 0.2959^2 \times 0]e^{-2\times0.08\times0.5} = 9.61$$

or \$9.61.

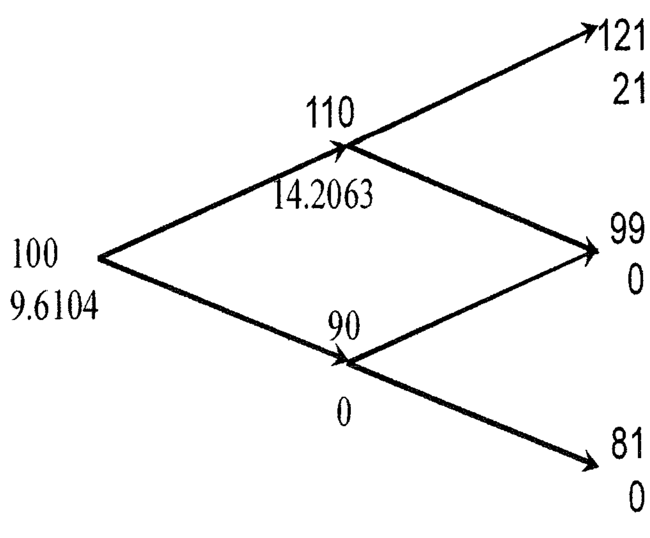


Figure S11.1 Tree for Problem 11.5

Problem 11.6.

For the situation considered in Problem 11.5, what is the value of a one-year European put option with a strike price of \$100? Verify that the European call and European put prices satisfy put–call parity.

Figure S11.2 shows how we can value the put option using the same tree as in Problem 11.5. The value of the option is \$1.92. The option value can also be calculated directly from equation (11.10):

$$e^{-2\times 0.08\times 0.5}[0.7041^2\times 0+2\times 0.7041\times 0.2959\times 1+0.2959^2\times 19]=1.92$$

or \$1.92. The stock price plus the put price is $100+1.92=\$101.92$. The present value of the strike price plus the call price is $100e^{-0.08\times 1}+9.61=\$101.92$. These are the same, verifying that put–call parity holds.

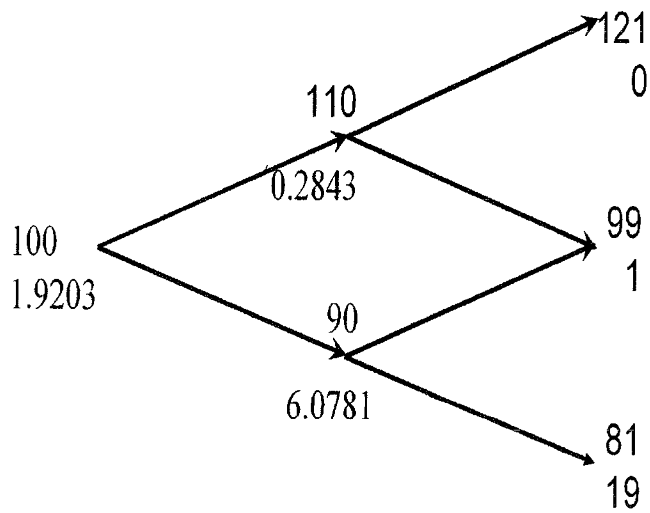


Figure S11.2 Tree for Problem 11.6

Problem 11.7.

What are the formulas for u and d in terms of volatility?

$u=e^{\sigma\sqrt{\Delta t}}$ and $d=e^{-\sigma\sqrt{\Delta t}}$

Problem 11.8.

Consider the situation in which stock price movements during the life of a European option are governed by a two-step binomial tree. Explain why it is not possible to set up a position in the stock and the option that remains riskless for the whole of the life of the option.

The riskless portfolio consists of a short position in the option and a long position in Δ shares. Because Δ changes during the life of the option, this riskless portfolio must also change.

Problem 11.9.

A stock price is currently $50. It is known that at the end of two months it will be either $53 or $48. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a two-month European call option with a strike price of $49? Use no-arbitrage arguments.

At the end of two months the value of the option will be either $4 (if the stock price is $53) or $0 (if the stock price is $48). Consider a portfolio consisting of:

$$\begin{array}{rcl} +\Delta & : & \text{shares} \\ -1 & : & \text{option} \end{array}$$

The value of the portfolio is either 48Δ or $53\Delta - 4$ in two months. If

$$48\Delta = 53\Delta - 4$$

i.e.,

$$\Delta = 0.8$$

the value of the portfolio is certain to be 38.4. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$0.8 \times 50 - f$$

where f is the value of the option. Since the portfolio must earn the risk-free rate of interest

$$(0.8 \times 50 - f)e^{0.10\times 2/12} = 38.4$$

i.e.,

$$f = 2.23$$

The value of the option is therefore $2.23.

This can also be calculated directly from equations (11.2) and (11.3). $u = 1.06$, $d = 0.96$ so that

$$p = \frac{e^{0.10\times 2/12} - 0.96}{1.06 - 0.96} = 0.5681$$

and

$$f = e^{-0.10\times 2/12} \times 0.5681 \times 4 = 2.23$$

Problem 11.10.

A stock price is currently \$80. It is known that at the end of four months it will be either \$75 or \$85. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a four-month European put option with a strike price of \$80? Use no-arbitrage arguments.

At the end of four months the value of the option will be either \$5 (if the stock price is \$75) or \$0 (if the stock price is \$85). Consider a portfolio consisting of:

$$\begin{array}{rcl} -\Delta & : & \text{shares} \\ +1 & : & \text{option} \end{array}$$

(Note: The delta, Δ of a put option is negative. We have constructed the portfolio so that it is +1 option and $-\Delta$ shares rather than -1 option and $+\Delta$ shares so that the initial investment is positive.)

The value of the portfolio is either -85Δ or $-75\Delta + 5$ in four months. If

$$-85\Delta = -75\Delta + 5$$

i.e.,

$$\Delta = -0.5$$

the value of the portfolio is certain to be 42.5. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$0.5 \times 80 + f$$

where f is the value of the option. Since the portfolio is riskless

$$(0.5 \times 80 + f)e^{0.05 \times 4/12} = 42.5$$

i.e.,

$$f = 1.80$$

The value of the option is therefore \$1.80.

This can also be calculated directly from equations (11.2) and (11.3). $u = 1.0625$, $d = 0.9375$ so that

$$p = \frac{e^{0.05 \times 4/12} - 0.9375}{1.0625 - 0.9375} = 0.6345$$

$1 - p = 0.3655$ and

$$f = e^{-0.05 \times 4/12} \times 0.3655 \times 5 = 1.80$$

Problem 11.11.

A stock price is currently \$40. It is known that at the end of three months it will be either \$45 or \$35. The risk-free rate of interest with quarterly compounding is 8% per annum. Calculate the value of a three-month European put option on the stock with *an exercise price of \$40. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answers.*

At the end of three months the value of the option is either \$5 (if the stock price is \$35) or \$0 (if the stock price is \$45).

Consider a portfolio consisting of:

$$\begin{array}{rcl} -\Delta & : & \text{shares} \\ +1 & : & \text{option} \end{array}$$

(Note: The delta, Δ, of a put option is negative. We have constructed the portfolio so that it is +1 option and $-\Delta$ shares rather than -1 option and $+\Delta$ shares so that the initial investment is positive.)

The value of the portfolio is either $-35\Delta + 5$ or -45Δ. If:

$$-35\Delta + 5 = -45\Delta$$

i.e.,

$$\Delta = -0.5$$

the value of the portfolio is certain to be 22.5. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is

$$-40\Delta + f$$

where f is the value of the option. Since the portfolio must earn the risk-free rate of interest

$$(40 \times 0.5 + f) \times 1.02 = 22.5$$

Hence

$$f = 2.06$$

i.e., the value of the option is \$2.06.

This can also be calculated using risk-neutral valuation. Suppose that p is the probability of an upward stock price movement in a risk-neutral world. We must have

$$45p + 35(1 - p) = 40 \times 1.02$$

i.e.,

$$10p = 5.8$$

or:

$$p = 0.58$$

The expected value of the option in a risk-neutral world is:

$$0 \times 0.58 + 5 \times 0.42 = 2.10$$

This has a present value of

$$\frac{2.10}{1.02} = 2.06$$

This is consistent with the no-arbitrage answer.

Problem 11.12.

A stock price is currently $50. Over each of the next two three-month periods it is expected to go up by 6% or down by 5%. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a six-month European call option with a strike price of $51?

A tree describing the behavior of the stock price is shown in Figure S11.3. The risk-neutral probability of an up move, p, is given by

$$p = \frac{e^{0.05\times 3/12} - 0.95}{1.06 - 0.95} = 0.5689$$

There is a payoff from the option of $56.18 - 51 = 5.18$ for the highest final node (which corresponds to two up moves) zero in all other cases. The value of the option is therefore

$$5.18 \times 0.5689^2 \times e^{-0.05\times 6/12} = 1.635$$

This can also be calculated by working back through the tree as indicated in Figure S11.3. The value of the call option is the lower number at each node in the figure.

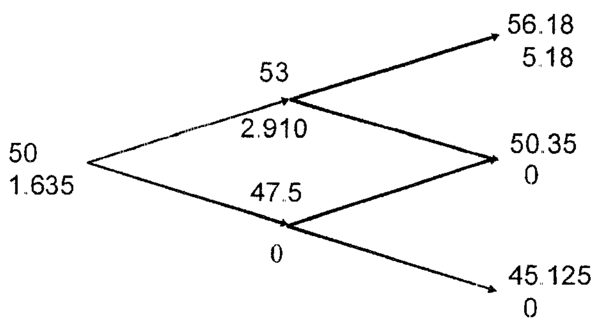


Figure S11.3 Tree for Problem 11.12

Problem 11.13.

For the situation considered in Problem 11.12, what is the value of a six-month European put option with a strike price of $51? Verify that the European call and European put prices satisfy put–call parity. If the put option were American, would it ever be optimal to exercise it early at any of the nodes on the tree?

The tree for valuing the put option is shown in Figure S11.4. We get a payoff of $51 - 50.35 = 0.65$ if the middle final node is reached and a payoff of $51 - 45.125 = 5.875$ if the lowest final node is reached. The value of the option is therefore

$$(0.65 \times 2 \times 0.5689 \times 0.4311 + 5.875 \times 0.4311^2)e^{-0.05\times 6/12} = 1.376$$

This can also be calculated by working back through the tree as indicated in Figure S11.4.

The value of the put plus the stock price is from Problem 11.12

$$1.376 + 50 = 51.376$$

The value of the call plus the present value of the strike price is

$$1.635 + 51e^{-0.05\times 6/12} = 51.376$$

This verifies that put–call parity holds

To test whether it worth exercising the option early we compare the value calculated for the option at each node with the payoff from immediate exercise. At node C the payoff from immediate exercise is $51 - 47.5 = 3.5$. Because this is greater than 2.8664, the option should be exercised at this node. The option should not be exercised at either node A or node B.

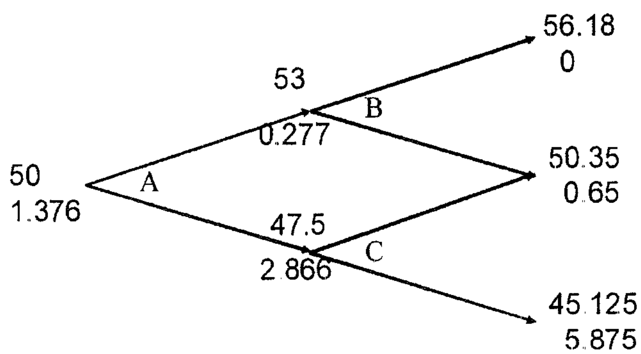


Figure S11.4 Tree for Problem 11.13

Problem 11.14.

A stock price is currently \$25. It is known that at the end of two months it will be either \$23 or \$27. The risk-free interest rate is 10% per annum with continuous compounding. Suppose S_T is the stock price at the end of two months. What is the value of a derivative that pays off S_T^2 at this time?

At the end of two months the value of the derivative will be either 529 (if the stock price is 23) or 729 (if the stock price is 27). Consider a portfolio consisting of:

$$\begin{array}{rcl} +\Delta & : & \text{shares} \\ -1 & : & \text{derivative} \end{array}$$

The value of the portfolio is either $27\Delta - 729$ or $23\Delta - 529$ in two months. If

$$27\Delta - 729 = 23\Delta - 529$$

i.e.,

$$\Delta = 50$$

the value of the portfolio is certain to be 621. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$50 \times 25 - f$$

where f is the value of the derivative. Since the portfolio must earn the risk-free rate of interest

$$(50 \times 25 - f)e^{0.10\times 2/12} = 621$$

i.e.,

$$f = 639.3$$

The value of the option is therefore \$639.3.

This can also be calculated directly from equations (11.2) and (11.3). $u = 1.08$, $d = 0.92$ so that

$$p = \frac{e^{0.10\times 2/12} - 0.92}{1.08 - 0.92} = 0.6050$$

and

$$f = e^{-0.10\times 2/12}(0.6050 \times 729 + 0.3950 \times 529) = 639.3$$

Problem 11.15.

Calculate u, d, and p when a binomial tree is constructed to value an option on a foreign currency. The tree step size is one month, the domestic interest rate is 5% per annum, the foreign interest rate is 8% per annum, and the volatility is 12% per annum.

In this case

$$a = e^{(0.05-0.08)\times 1/12} = 0.9975$$

$$u = e^{0.12\sqrt{1/12}} = 1.0352$$

$$d = 1/u = 0.9660$$

$$p = \frac{0.9975 - 0.9660}{1.0352 - 0.9660} = 0.4553$$

ASSIGNMENT QUESTIONS

Problem 11.16.

A stock price is currently \$50. It is known that at the end of six months it will be either \$60 or \$42. The risk-free rate of interest with continuous compounding is 12% per annum. Calculate the value of a six-month European call option on the stock with an exercise price of \$48. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answers.

At the end of six months the value of the option will be either \$12 (if the stock price is \$60) or \$0 (if the stock price is \$42). Consider a portfolio consisting of:

$$\begin{array}{rcl} +\Delta & : & \text{shares} \\ -1 & : & \text{option} \end{array}$$

The value of the portfolio is either 42Δ or $60\Delta - 12$ in six months. If

$$42\Delta = 60\Delta - 12$$

i.e.,

$$\Delta = 0.6667$$

the value of the portfolio is certain to be 28. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$0.6667 \times 50 - f$$

where f is the value of the option. Since the portfolio must earn the risk-free rate of interest

$$(0.6667 \times 50 - f)e^{0.12 \times 0.5} = 28$$

i.e.,

$$f = 6.96$$

The value of the option is therefore \$6.96.

This can also be calculated using risk-neutral valuation. Suppose that p is the probability of an upward stock price movement in a risk-neutral world. We must have

$$60p + 42(1 - p) = 50 \times e^{0.06}$$

i.e.,

$$18p = 11.09$$

or:

$$p = 0.6161$$

The expected value of the option in a risk-neutral world is:

$$12 \times 0.6161 + 0 \times 0.3839 = 7.3932$$

This has a present value of

$$7.3932e^{-0.06} = 6.96$$

Hence the above answer is consistent with risk-neutral valuation.

Problem 11.17.

A stock price is currently \$40. Over each of the next two three-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 12% per annum with continuous compounding.

a. *What is the value of a six-month European put option with a strike price of \$42?*
b. *What is the value of a six-month American put option with a strike price of \$42?*

(a) A tree describing the behavior of the stock price is shown in Figure M11.1. The risk-neutral probability of an up move, p, is given by

$$p = \frac{e^{0.12\times 3/12} - 0.90}{1.1 - 0.9} = 0.6523$$

Calculating the expected payoff and discounting, we obtain the value of the option as

$$[2.4 \times 2 \times 0.6523 \times 0.3477 + 9.6 \times 0.3477^2]e^{-0.12\times 6/12} = 2.118$$

The value of the European option is 2.118. This can also be calculated by working back through the tree as shown in Figure M11.1. The second number at each node is the value of the European option.

(b) The value of the American option is shown as the third number at each node on the tree. It is 2.537. This is greater than the value of the European option because it is optimal to exercise early at node C.

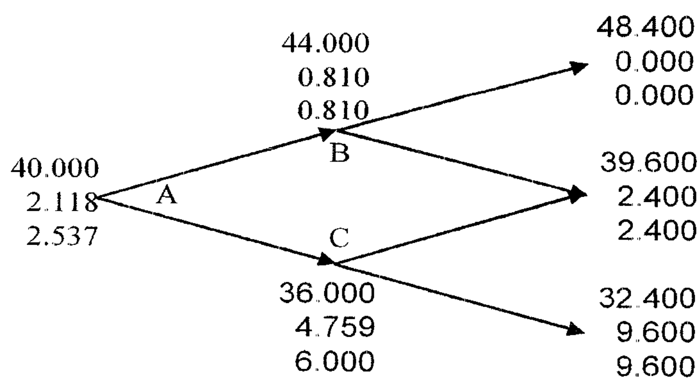


Figure M11.1 Tree to evaluate European and American put options in Problem 11.17. At each node, upper number is the stock price; next number is the European put price; final number is the American put price

Problem 11.18.

Using a "trial-and-error" approach, estimate how high the strike price has to be in Problem 11.17 for it to be optimal to exercise the option immediately.

Trial and error shows that immediate early exercise is optimal when the strike price is above 43.2.

This can be also shown to be true algebraically. Suppose the strike price increases by a relatively small amount q. This increases the value of being at node C by q and the value of being at node B by $0.3477e^{-0.03}q = 0.3374q$. It therefore increases the value of being at node A by

$$(0.6523 \times 0.3374q + 0.3477q)e^{-0.03} = 0.551q$$

For early exercise at node A we require $2.537+0.551q < 2+q$ or $q > 1.196$. This corresponds to the strike price being greater than 43.196.

Problem 11.19.

A stock price is currently \$30. During each two-month period for the next four months it is expected to increase by 8% or reduce by 10%. The risk-free interest rate is 5%. Use a two-step tree to calculate the value of a derivative that pays off $[\max(30 - S_T, 0]^2$ *where* S_T *is the stock price in four months? If the derivative is American-style, should it be exercised early?*

This type of option is known as a power option. A tree describing the behavior of the stock price is shown in Figure M11.2. The risk-neutral probability of an up move, p, is given by

$$p = \frac{e^{0.05\times 2/12} - 0.9}{1.08 - 0.9} = 0.6020$$

Calculating the expected payoff and discounting, we obtain the value of the option as

$$[0.7056 \times 2 \times 0.6020 \times 0.3980 + 32.49 \times 0.3980^2]e^{-0.05\times 4/12} = 5.394$$

The value of the European option is 5.394. This can also be calculated by working back through the tree as shown in Figure M11.2. The second number at each node is the value of the European option.

Early exercise at node C would give 9.0 which is less than 13.2449. The option should therefore not be exercised early if it is American.

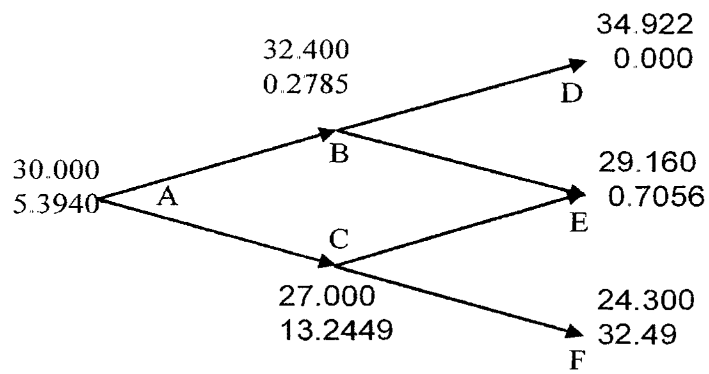


Figure M11.2 Tree to evaluate European power option in Problem 11.19. At each node, upper number is the stock price; next number is the option price

Problem 11.20.

Consider a European call option on a non-dividend-paying stock where the stock price is $40, the strike price is $40, the risk-free rate is 4% per annum, the volatility is 30% per annum, and the time to maturity is six months.

a. Calculate u, d, and p for a two step tree
b. Value the option using a two step tree.
c. Verify that DerivaGem gives the same answer
d. Use DerivaGem to value the option with 5, 50, 100, and 500 time steps.

(a) This problem is based on the material in Section 11.8. In this case $\Delta t = 0.25$ so that $u = e^{0.30\times\sqrt{0.25}} = 1.1618$, $d = 1/u = 0.8607$, and

$$p = \frac{e^{0.04\times 0.25} - 0.8607}{1.1618 - 0.8607} = 0.4959$$

(b) and (c) The value of the option using a two-step tree as given by DerivaGem is shown in Figure M11.3 to be 3.3739. To use DerivaGem choose the first worksheet, select Equity as the underlying type, and select Binomial European as the Option Type. After carrying out the calculations select Display Tree.

(d) With 5, 50, 100, and 500 time steps the value of the option is 3.9229, 3.7394, 3.7478, and 3.7545, respectively.

At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Values in red are a result of early exercise.

Strike price = 40
Discount factor per step = 0.9900
Time step, dt = 0.2500 years, 91.25 days
Growth factor per step, a = 1.0101
Probability of up move, p = 0.4959
Up step size, u = 1.1618
Down step size, d = 0.8607

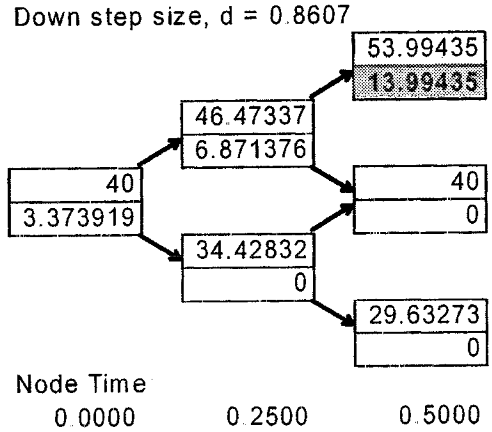


Figure M11.3 Tree produced by DerivaGem to evaluate European option in Problem 11.20.

Problem 11.21.

Repeat Problem 11.20 for an American put option on a futures contract. The strike price and the futures price are $50, the risk-free rate is 10%, the time to maturity is six months, and the volatility is 40% per annum.

(a) In this case $\Delta t = 0.25$ and $u = e^{0.40\times\sqrt{0.25}} = 1.2214$, $d = 1/u = 0.8187$, and

$$p = \frac{e^{0.1\times 0.25} - 0.8187}{1.2214 - 0.8187} = 0.4502$$

(b) and (c) The value of the option using a two-step tree is 4.8604.

(d) With 5, 50, 100, and 500 time steps the value of the option is 5.6858, 5.3869, 5.3981, and 5.4072, respectively.

Problem 11.22.

Footnote 1 shows that the correct discount rate to use for the real world expected payoff in the case of the call option considered in Figure 11.1 is 42.6%. Show that if the option is a put rather than a call the discount rate is –52.5%. Explain why the two real-world discount rates are so different.

The value of the put option is

$$(0.6523 \times 0 + 0.3477 \times 3)e^{-0.12 \times 3/12} = 1.0123$$

The expected payoff in the real world is

$$(0.7041 \times 0 + 0.2959 \times 3) = 0.8877$$

The discount rate R that should be used in the real world is therefore given by solving

$$1.0123 = 0.8877e^{-0.25R}$$

The solution to this is -0.525 or 52.5%.

The call option has a high positive discount rate because it has high positive systematic risk. The put option has a high negative discount rate because it has high negative systematic risk.

CHAPTER 12
Wiener processes and Itô's Lemma

Notes for the Instructor

The chapter provides some basic knowledge about Wiener processes, develops the geometric Brownian motion model of stock price behavior, and covers Itô's lemma. The book has been designed so that this chapter can be skipped if the instructor considers it too technical. For example, a popular way of using the book for a first MBA elective in derivatives is to assign the first 18 chapters, excluding Chapter 12 and Section 13.6 of Chapter 13.

I find that most students have surprisingly little difficulty with the material in Chapter 12. I usually start by discussing the distinction between continuous time and discrete time stochastic processes and the distinction between continuous variable and discrete variable stochastic processes. I do this with simple examples of models of stock price movements. A discrete time, discrete variable model would be one where every day a stock price has a probability p_1 of moving up by \$1, a probability p_2 of remaining the same, and a probability p_3 of moving down by \$1. A continuous time, discrete variable model would be one where price changes of \$1 are generated by a Poisson process. A discrete time, continuous variable model would be one where in each small time interval, stock price movements are sampled from a continuous distribution. The main purpose of the chapter is to develop a continuous time, continuous variable model as a limiting case of this last model.

The nature of Markov processes and the fact that they are consistent with market efficiency needs to be explained carefully. I find it useful to discuss Problems 12.1 and 12.2 in class.

I explain Wiener processes by starting with a discrete time, continuous variable model where values of the variable are observed every year. I assume that the change in the value of the variable during the year is a random sample from the distribution $\phi(0, 1)$. (Note that the second argument of ϕ is the variance not the standard deviation.) I then suppose that values of the variable are observed every 6 months and ask what distribution for 6-month changes is consistent with the distribution of 1-year changes. The answer is $\phi(0, \frac{1}{2})$. When 3-month changes are considered, the distribution is $\phi(0, 0.25)$. When time intervals of length Δt are considered, the distribution is $\phi(0, \Delta t)$. A random sample from $\phi(0, \Delta t)$ is $\epsilon\sqrt{\Delta t}$. This explains where the definition of a Wiener process comes from. In particular it explains why we take the square root of time. Once Wiener processes are understood, generalized Wiener processes and Itô processes should not cause too much difficulty.

In understanding a stochastic process, I find that an explanation of how it can be simulated is useful and usually go through an example such as that in Section 12.3.

The amount of time spent on Itô's lemma will depend on the mathematical backgrounds of students. Mathematically inclined students generally feel quite a sense of achievement when they have managed to work through the material in the appendix to Chapter 12.

All of Problems 12.12 to 12.16 work well as assignment questions.

QUESTIONS AND PROBLEMS

Problem 12.1.

What would it mean to assert that the temperature at a certain place follows a Markov process? Do you think that temperatures do, in fact, follow a Markov process?

Imagine that you have to forecast the future temperature from a) the current temperature, b) the history of the temperature in the last week, and c) a knowledge of seasonal averages and seasonal trends. If temperature followed a Markov process, the history of the temperature in the last week would be irrelevant.

To answer the second part of the question you might like to consider the following scenario for the first week in May:

(i) Monday to Thursday are warm days; today, Friday, is a very cold day.

(ii) Monday to Friday are all very cold days.

What is your forecast for the weekend? If you are more pessimistic in the case of the second scenario, temperatures do not follow a Markov process.

Problem 12.2.

Can a trading rule based on the past history of a stock's price ever produce returns that are consistently above average? Discuss.

The first point to make is that any trading strategy can, just because of good luck, produce above average returns. The key question is whether a trading strategy *consistently* outperforms the market when adjustments are made for risk. It is certainly possible that a trading strategy could do this. However, when enough investors know about the strategy and trade on the basis of the strategy, the profit will disappear.

As an illustration of this, consider a phenomenon known as the small firm effect. Portfolios of stocks in small firms appear to have outperformed portfolios of stocks in large firms when appropriate adjustments are made for risk. Papers were published about this in the early 1980s and mutual funds were set up to take advantage of the phenomenon. There is some evidence that this has resulted in the phenomenon disappearing.

Problem 12.3.

A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.5 per quarter and a variance rate of 4.0 per quarter. How high does the company's initial cash position have to be for the company to have a less than 5% chance of a negative cash position by the end of one year?

Suppose that the company's initial cash position is x. The probability distribution of the cash position at the end of one year is

$$\phi(x + 4 \times 0.5, 4 \times 4) = \phi(x + 2.0,\ 16)$$

where $\phi(m, v)$ is a normal probability distribution with mean m and variance v. The probability of a negative cash position at the end of one year is

$$N\left(-\frac{x+2.0}{4}\right)$$

where $N(x)$ is the cumulative probability that a standardized normal variable (with mean zero and standard deviation 1.0) is less than x. From normal distribution tables

$$N\left(-\frac{x+2.0}{4}\right) = 0.05$$

when:

$$-\frac{x+2.0}{4} = -1.6449$$

i.e., when $x = 4.5796$. The initial cash position must therefore be $4.58 million.

Problem 12.4.

Variables X_1 and X_2 follow generalized Wiener processes with drift rates μ_1 and μ_2 and variances σ_1^2 and σ_2^2. What process does $X_1 + X_2$ follow if:

(a) The changes in X_1 and X_2 in any short interval of time are uncorrelated?

(b) There is a correlation ρ between the changes in X_1 and X_2 in any short interval of time?

(a) Suppose that X_1 and X_2 equal a_1 and a_2 initially. After a time period of length T, X_1 has the probability distribution

$$\phi(a_1 + \mu_1 T,\ \sigma_1^2 T)$$

and X_2 has a probability distribution

$$\phi(a_2 + \mu_2 T,\ \sigma_2^2 T)$$

From the property of sums of independent normally distributed variables, $X_1 + X_2$ has the probability distribution

$$\phi\left(a_1 + \mu_1 T + a_2 + \mu_2 T,\ \sigma_1^2 T + \sigma_2^2 T\right)$$

i.e.,

$$\phi\left[a_1 + a_2 + (\mu_1 + \mu_2)T, (\sigma_1^2 + \sigma_2^2)T\right]$$

This shows that $X_1 + X_2$ follows a generalized Wiener process with drift rate $\mu_1 + \mu_2$ and variance rate $\sigma_1^2 + \sigma_2^2$.

(b) In this case the change in the value of $X_1 + X_2$ in a short interval of time Δt has the probability distribution:

$$\phi\left[(\mu_1 + \mu_2)\Delta t, (\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)\Delta t\right]$$

If μ_1, μ_2, σ_1, σ_2 and ρ are all constant, arguments similar to those in Section 12.2 show that the change in a longer period of time T is

$$\phi\left[(\mu_1+\mu_2)T, (\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)T\right]$$

The variable, X_1+X_2, therefore follows a generalized Wiener process with drift rate $\mu_1+\mu_2$ and variance rate $\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2$.

Problem 12.5.

Consider a variable, S, that follows the process

$$dS = \mu\, dt + \sigma\, dz$$

For the first three years, $\mu = 2$ and $\sigma = 3$; for the next three years, $\mu = 3$ and $\sigma = 4$. If the initial value of the variable is 5, what is the probability distribution of the value of the variable at the end of year six?

The change in S during the first three years has the probability distribution

$$\phi(2\times 3,\ 9\times 3) = \phi(6,\ 27)$$

The change during the next three years has the probability distribution

$$\phi(3\times 3,\ 16\times 3) = \phi(9,\ 48)$$

The change during the six years is the sum of a variable with probability distribution $\phi(6,\ 27)$ and a variable with probability distribution $\phi(9,\ 48)$. The probability distribution of the change is therefore

$$\phi(6+9, 27+48)$$
$$= \phi(15,\ 75)$$

Since the initial value of the variable is 5, the probability distribution of the value of the variable at the end of year six is

$$\phi(20,\ 75)$$

Problem 12.6.

Suppose that G is a function of a stock price, S and time. Suppose that σ_S and σ_G are the volatilities of S and G. Show that when the expected return of S increases by $\lambda\sigma_S$, the growth rate of G increases by $\lambda\sigma_G$, where λ is a constant.

From Itô's lemma

$$\sigma_G G = \frac{\partial G}{\partial S}\sigma_S S$$

Also the drift of G is

$$\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2$$

where μ is the expected return on the stock. When μ increases by $\lambda\sigma_S$, the drift of G increases by

$$\frac{\partial G}{\partial S}\lambda\sigma_S S$$

or

$$\lambda\sigma_G G$$

The growth rate of G, therefore, increases by $\lambda\sigma_G$.

Problem 12.7.

Stock A and stock B both follow geometric Brownian motion. Changes in any short interval of time are uncorrelated with each other. Does the value of a portfolio consisting of one of stock A and one of stock B follow geometric Brownian motion? Explain your answer.

Define S_A, μ_A and σ_A as the stock price, expected return and volatility for stock A. Define S_B, μ_B and σ_B as the stock price, expected return and volatility for stock B. Define ΔS_A and ΔS_B as the change in S_A and S_B in time Δt. Since each of the two stocks follows geometric Brownian motion,

$$\Delta S_A = \mu_A S_A \Delta t + \sigma_A S_A \epsilon_A \sqrt{\Delta t}$$

$$\Delta S_B = \mu_B S_B \Delta t + \sigma_B S_B \epsilon_B \sqrt{\Delta t}$$

where ϵ_A and ϵ_B are independent random samples from a normal distribution.

$$\Delta S_A + \Delta S_B = (\mu_A S_A + \mu_B S_B)\Delta t + (\sigma_A S_A \epsilon_A + \sigma_B S_B \epsilon_B)\sqrt{\Delta t}$$

This *cannot* be written as

$$\Delta S_A + \Delta S_B = \mu(S_A + S_B)\Delta t + \sigma(S_A + S_B)\epsilon\sqrt{\Delta t}$$

for any constants μ and σ. (Neither the drift term nor the stochastic term correspond.) Hence the value of the portfolio does not follow geometric Brownian motion.

Problem 12.8.

The process for the stock price in equation (12.8) is

$$\Delta S = \mu S\,\Delta t + \sigma S\epsilon\,\sqrt{\Delta t}$$

where μ and σ are constant. Explain carefully the difference between this model and each of the following:

$$\Delta S = \mu\,\Delta t + \sigma\epsilon\,\sqrt{\Delta t}$$
$$\Delta S = \mu S\,\Delta t + \sigma\epsilon\,\sqrt{\Delta t}$$
$$\Delta S = \mu\,\Delta t + \sigma S\epsilon\,\sqrt{\Delta t}$$

Why is the model in equation (12.8) a more appropriate model of stock price behavior than any of these three alternatives?

In:

$$\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

the expected increase in the stock price and the variability of the stock price are constant when both are expressed as a proportion (or as a percentage) of the stock price

In:

$$\Delta S = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}$$

the expected increase in the stock price and the variability of the stock price are constant in absolute terms. For example, if the expected growth rate is \$5 per annum when the stock price is \$25, it is also \$5 per annum when it is \$100. If the standard deviation of weekly stock price movements is \$1 when the price is \$25, it is also \$1 when the price is \$100.

In:

$$\Delta S = \mu S \Delta t + \sigma \epsilon \sqrt{\Delta t}$$

the expected increase in the stock price is a constant proportion of the stock price while the variability is constant in absolute terms.

In:

$$\Delta S = \mu \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

the expected increase in the stock price is constant in absolute terms while the variability of the proportional stock price change is constant.

The model:

$$\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

is the most appropriate one since it is most realistic to assume that the expected *percentage return* and the variability of the *percentage return* in a short interval are constant.

Problem 12.9.

It has been suggested that the short-term interest rate, r, follows the stochastic process

$$dr = a(b - r)\,dt + rc\,dz$$

where a, b, and c are positive constants and dz is a Wiener process. Describe the nature of this process.

The drift rate is $a(b - r)$. Thus, when the interest rate is above b the drift rate is negative and, when the interest rate is below b, the drift rate is positive. The interest rate is therefore continually pulled towards the level b. The rate at which it is pulled toward this level is a. A volatility equal to c is superimposed upon the "pull" or the drift.

Suppose $a = 0.4$, $b = 0.1$ and $c = 0.15$ and the current interest rate is 20% per annum. The interest rate is pulled towards the level of 10% per annum. This can be regarded as a long run average. The current drift is -4% per annum so that the expected rate at the end of one year is about 16% per annum. (In fact it is slightly greater than this, because as the interest rate decreases, the "pull" decreases.) Superimposed upon the drift is a volatility of 15% per annum.

Problem 12.10.

Suppose that a stock price, S, follows geometric Brownian motion with expected return μ and volatility σ:

$$dS = \mu S\, dt + \sigma S\, dz$$

What is the process followed by the variable S^n? Show that S^n also follows geometric Brownian motion.

If $G(S,t) = S^n$ then $\partial G/\partial t = 0$, $\partial G/\partial S = nS^{n-1}$, and $\partial^2 G/\partial S^2 = n(n-1)S^{n-2}$. Using Itô's lemma:

$$dG = [\mu nG + \frac{1}{2}n(n-1)\sigma^2 G]\, dt + \sigma nG\, dz$$

This shows that $G = S^n$ follows geometric Brownian motion where the expected return is

$$\mu n + \frac{1}{2}n(n-1)\sigma^2$$

and the volatility is $n\sigma$. The stock price S has an expected return of μ and the expected value of S_T is $S_0 e^{\mu T}$. The expected value of S_T^n is

$$S_0^n e^{[\mu n + \frac{1}{2}n(n-1)\sigma^2]T}$$

Problem 12.11.

Suppose that x is the yield to maturity with continuous compounding on a zero-coupon bond that pays off \$1 at time T. Assume that x follows the process

$$dx = a(x_0 - x)\, dt + sx\, dz$$

where a, x_0, and s are positive constants and dz is a Wiener process. What is the process followed by the bond price?

The process followed by B, the bond price, is from Itô's lemma:

$$dB = \left[\frac{\partial B}{\partial x}a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}s^2x^2\right] dt + \frac{\partial B}{\partial x}sxdz$$

Since:

$$B = e^{-x(T-t)}$$

the required partial derivatives are

$$\frac{\partial B}{\partial t} = xe^{-x(T-t)} = xB$$
$$\frac{\partial B}{\partial x} = -(T-t)e^{-x(T-t)} = -(T-t)B$$
$$\frac{\partial^2 B}{\partial x^2} = (T-t)^2 e^{-x(T-t)} = (T-t)^2 B$$

Hence:

$$dB = \left[-a(x_0 - x)(T-t) + x + \frac{1}{2}s^2x^2(T-t)^2\right] Bdt - sx(T-t)Bdz$$

ASSIGNMENT QUESTIONS

Problem 12.12.

Suppose that a stock price has an expected return of 16% per annum and a volatility of 30% per annum. When the stock price at the end of a certain day is \$50, calculate the following:

(a) The expected stock price at the end of the next day.
(b) The standard deviation of the stock price at the end of the next day.
(c) The 95% confidence limits for the stock price at the end of the next day.

With the notation in the text

$$\frac{\Delta S}{S} \sim \phi(\mu \Delta t,\ \sigma^2 \Delta t)$$

In this case $S = 50$, $\mu = 0.16$, $\sigma = 0.30$ and $\Delta t = 1/365 = 0.00274$. Hence

$$\begin{aligned}\frac{\Delta S}{50} &\sim \phi(0.16 \times 0.00274,\ 0.09 \times 0.00274)\\ &= \phi(0.00044,\ 0.000247)\end{aligned}$$

and

$$\Delta S \sim \phi(50 \times 0.00044,\ 50^2 \times 0.000247)$$

that is,

$$\Delta S \sim \phi(0.022,\ 0.6164)$$

(a) The expected stock price at the end of the next day is therefore 50.022
(b) The standard deviation of the stock price at the end of the next day is $\sqrt{0.6154} = 0.785$
(c) 95% confidence limits for the stock price at the end of the next day are

$$50.022 - 1.96 \times 0.785 \quad \text{and} \quad 50.022 + 1.96 \times 0.785$$

i.e.,

$$48.48 \quad \text{and} \quad 51.56$$

Note that some students may consider one trading day rather than one calendar day. Then $\Delta t = 1/252 = 0.00397$. The answer to (a) is then 50.032. The answer to (b) is 0.945. The answers to part (c) are 48.18 and 51.88.

Problem 12.13.

A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.1 per month and a variance rate of 0.16 per month. The initial cash position is 2.0.

(a) What are the probability distributions of the cash position after one month, six months, and one year?

(b) What are the probabilities of a negative cash position at the end of six months and one year?

(c) At what time in the future is the probability of a negative cash position greatest?

(a) The probability distributions are:

$$\phi(2.0 + 0.1,\ 0.16) = \phi(2.1, 0.16)$$

$$\phi(2.0 + 0.6,\ 0.16 \times 6) = \phi(2.6,\ 0.96)$$

$$\phi(2.0 + 1.2,\ 0.16 \times 12) = \phi(3.2,\ 1.96)$$

(b) The chance of a random sample from $\phi(2.6,\ 0.96)$ being negative is

$$N\left(-\frac{2.6}{\sqrt{0.96}}\right) = N(-2.65)$$

where $N(x)$ is the cumulative probability that a standardized normal variable [i.e., a variable with probability distribution $\phi(0,\ 1)$] is less than x. From normal distribution tables $N(-2.65) = 0.0040$. Hence the probability of a negative cash position at the end of six months is 0.40%.

Similarly the probability of a negative cash position at the end of one year is

$$N\left(-\frac{3.2}{\sqrt{1.96}}\right) = N(-2.30) = 0.0107$$

or 1.07%.

(c) In general the probability distribution of the cash position at the end of x months is

$$\phi(2.0 + 0.1x,\ 0.16x)$$

The probability of the cash position being negative is maximized when:

$$\frac{2.0 + 0.1x}{\sqrt{0.16x}}$$

is minimized. Define

$$y = \frac{2.0 + 0.1x}{0.4\sqrt{x}} = 5x^{-\frac{1}{2}} + 0.25x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = -2.5x^{-\frac{3}{2}} + 0.125x^{-\frac{1}{2}}$$

$$= x^{-\frac{3}{2}}(-2.5 + 0.125x)$$

This is zero when $x = 20$ and it is easy to verify that $d^2y/dx^2 > 0$ for this value of x. It therefore gives a minimum value for y. Hence the probability of a negative cash position is greatest after 20 months.

Problem 12.14.

Suppose that x is the yield on a perpetual government bond that pays interest at the rate of \$1 per annum. Assume that x is expressed with continuous compounding, that interest is paid continuously on the bond, and that x follows the process

$$dx = a(x_0 - x)\,dt + sx\,dz$$

where a, x_0, and s are positive constants and dz is a Wiener process. What is the process followed by the bond price? What is the expected instantaneous return (including interest and capital gains) to the holder of the bond?

The process followed by B, the bond price, is from Itô's lemma:

$$dB = \left[\frac{\partial B}{\partial x}a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}s^2x^2\right]dt + \frac{\partial B}{\partial x}sxdz$$

In this case

$$B = \frac{1}{x}$$

so that:

$$\frac{\partial B}{\partial t} = 0; \quad \frac{\partial B}{\partial x} = -\frac{1}{x^2}; \quad \frac{\partial^2 B}{\partial x^2} = \frac{2}{x^3}$$

Hence

$$\begin{aligned} dB &= \left[-a(x_0 - x)\frac{1}{x^2} + \frac{1}{2}s^2x^2\frac{2}{x^3}\right]dt - \frac{1}{x^2}sxdz \\ &= \left[-a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}\right]dt - \frac{s}{x}dz \end{aligned}$$

The expected instantaneous rate at which capital gains are earned from the bond is therefore:

$$-a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}$$

The expected interest per unit time is 1. The total expected instantaneous return is therefore:

$$1 - a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}$$

When expressed as a proportion of the bond price this is:

$$\left(1 - a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}\right) \Big/ \left(\frac{1}{x}\right)$$

$$= x - \frac{a}{x}(x_0 - x) + s^2$$

Problem 12.15.

If S follows the geometric Brownian motion process in equation (12.6), what is the process followed by

a. $y = 2S$
b. $y = S^2$
c. $y = e^S$
d. $y = \frac{e^{r(T-t)}}{S}$

In each case express the coefficients of dt and dz in terms of y rather than S.

(a) In this case $\partial y/\partial S = 2$, $\partial^2 y/\partial S^2 = 0$, and $\partial y/\partial t = 0$ so that Itô's lemma gives

$$dy = 2\mu S\, dt + 2\sigma S\, dz$$

or

$$dy = \mu y\, dt + \sigma y\, dz$$

(b) In this case $\partial y/\partial S = 2S$, $\partial^2 y/\partial S^2 = 2$, and $\partial y/\partial t = 0$ so that Itô's lemma gives

$$dy = (2\mu S^2 + \sigma^2 S^2)\, dt + 2\sigma S^2\, dz$$

or

$$dy = (2\mu + \sigma^2) y\, dt + 2\sigma y\, dz$$

(c) In this case $\partial y/\partial S = e^S$, $\partial^2 y/\partial S^2 = e^S$, and $\partial y/\partial t = 0$ so that Itô's lemma gives

$$dy = (\mu S e^S + \sigma^2 S^2 e^S/2)\, dt + \sigma S e^S\, dz$$

or

$$dy = [\mu y \ln y + \sigma^2 y (\ln y)^2/2]\, dt + \sigma y \ln y\, dz$$

(d) In this case $\partial y/\partial S = -e^{r(T-t)}/S^2 = -y/S$, $\partial^2 y/\partial S^2 = 2e^{r(T-t)}/S^3 = 2y/S^2$, and $\partial y/\partial t = -re^{r(T-t)}/S = -ry$ so that Itô's lemma gives

$$dy = (-ry - \mu y + \sigma^2 y)\, dt - \sigma y\, dz$$

or

$$dy = -(r + \mu - \sigma^2) y\, dt - \sigma y\, dz$$

Problem 12.16.

A stock price is currently 50. Its expected return and volatility are 12% and 30%, respectively. What is the probability that the stock price will be greater than 80 in two years? (Hint $S_T > 80$ when $\ln S_T > \ln 80$.)

The variable $\ln S_T$ is normally distributed with mean $\ln S_0 + (\mu - \sigma^2/2)T$ and standard deviation $\sigma\sqrt{T}$. In this case $S_0 = 50$, $\mu = 0.12$, $T = 2$, and $\sigma = 0.30$ so that the mean and standard deviation of $\ln S_T$ are $\ln 50 + (0.12 - 0.3^2/2)2 = 4.062$ and $0.3\sqrt{2} = 0.424$, respectively. Also, $\ln 80 = 4.382$. The probability that $S_T > 80$ is the same as the probability that $\ln S_T > 4.382$. This is

$$1 - N\left(\frac{4.382 - 4.062}{0.424}\right) = 1 - N(0.754)$$

where $N(x)$ is the probability that a normally distributed variable with mean zero and standard deviation 1 is less than x. From the tables at the back of the book $N(0.754) = 0.775$ so that the required probability is 0.225.

CHAPTER 13
The Black–Scholes–Merton Model

Notes for the Instructor

This chapter covers important material: the lognormality of stock prices, the calculation of volatility from historical data, the Black–Scholes–Merton differential equation, risk-neutral valuation, the Black–Scholes–Merton option pricing formulas, implied volatilities, and the impact of dividends. Section 13.6 should be skipped if Chapter 12 has not already been covered.

The distinction between

μ: the expected rate of return in a short period of time, and

$\mu - \sigma^2/2$: the expected continuously compounded rate of return over any period of time

usually causes some problems. I have tried a few different approaches and think that the one that is now in the text works reasonably well.

Business Snapshot 13.2 on the causes of volatility generally leads to a lively discussion. I find that students have an easier time than academics in accepting that trading itself causes volatility!

When presenting Black–Scholes–Merton arguments I point out that in any small interval of time Δt, the stock price and the option price are perfectly correlated. This is the same as saying that the ratio $\Delta c/\Delta S$ is constant where Δc and ΔS are the change in c and S in time Δt respectively. It is possible to set up a portfolio consisting of a position in the derivative and a position in the stock which is, for the next small interval of time Δt, riskless. (For example, if

$$\frac{\Delta c}{\Delta S} = 0.4$$

a short position in 100 of the derivative security when combined with a long position in 40 of the stock is riskless for time Δt.) This is essentially what Black, Scholes, and Merton did to derive their differential equation. After presenting the Black–Scholes–Merton differential equation I like to got through Example 13.5 on forward contracts in class. Later the same example can be used to illustrate risk-neutral valuation.

The risk-neutral valuation argument must be covered carefully. It cannot be emphasized often enough that we are not assuming risk neutrality. It just happens that the value of a derivative security is independent of risk preferences.

When the Black–Scholes–Merton equation for pricing a call option is presented, students sometimes ask for the intuition behind it and are frustrated that they cannot easily derive it. I point out that a European call option holder gets $S_T - K$ whenever $S_T > K$. This means that the option holder is long a security that pays off S_T when $S_T > K$ and short a security that pays off K when $S_T > K$. The first security is known as an asset-or-nothing call. The second security is known as a cash-or-nothing call. The probability that $S_T > K$ in a risk-neutral world is $N(d_2)$. (See Problem 13.22). The expected payoff from the second security in a risk-neutral world is therefore $KN(d_2)$. From risk-neutral valuation, the value of the security is $KN(d_2)e^{-rT}$. The value of the first security can also be calculated using risk-neutral valuation. It turns out to be $S_0N(d_1)$. (Students have to take this on faith). Putting the two results together we get the Black–Scholes–Merton formula for a European call option.

When calculating the cumulative normal distribution function, most students will choose to use the table at the end of the book or the Excel function NORMSDIST. The polynomial approximation may be useful if they choose to write their own software. I encourage students to develop their own Excel worksheets for option pricing as well as using DerivaGem.

All the assignment questions work well. My favorite is 13.28.

QUESTIONS AND PROBLEMS

Problem 13.1.

What does the Black–Scholes–Merton stock option pricing model assume about the probability distribution of the stock price in one year? What does it assume about the continuously compounded rate of return on the stock during the year?

The Black–Scholes–Merton option pricing model assumes that the probability distribution of the stock price in 1 year (or at any other future time) is lognormal. It assumes that the continuously compounded rate of return on the stock during the year is normally distributed.

Problem 13.2.

The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

The standard deviation of the percentage price change in time Δt is $\sigma\sqrt{\Delta t}$ where σ is the volatility. In this problem $\sigma = 0.3$ and, assuming 252 trading days in one year, $\Delta t = 1/252 = 0.004$ so that $\sigma\sqrt{\Delta t} = 0.3\sqrt{0.004} = 0.019$ or 1.9%.

Problem 13.3.

Explain the principle of risk-neutral valuation.

The price of an option or other derivative when expressed in terms of the price of the underlying stock is independent of risk preferences. Options therefore have the same value in a risk-neutral world as they do in the real world. We may therefore assume that the world is risk neutral for the purposes of valuing options. This simplifies the analysis. In a risk-neutral world all securities have an expected return equal to risk-free interest rate. Also, in a risk-neutral world, the appropriate discount rate to use for expected future cash flows is the risk-free interest rate.

Problem 13.4.

Calculate the price of a three-month European put option on a non-dividend-paying stock with a strike price of \$50 when the current stock price is \$50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum.

In this case $S_0 = 50$, $K = 50$, $r = 0.1$, $\sigma = 0.3$, $T = 0.25$, and

$$d_1 = \frac{\ln(50/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.2417$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = 0.0917$$

The European put price is

$$50N(-0.0917)e^{-0.1\times0.25} - 50N(-0.2417)$$

$$= 50 \times 0.4634e^{-0.1\times0.25} - 50 \times 0.4045 = 2.37$$

or \$2.37.

Problem 13.5.

What difference does it make to your calculations in Problem 13.4 if a dividend of \$1.50 is expected in two months?

In this case we must subtract the present value of the dividend from the stock price before using Black–Scholes. Hence the appropriate value of S_0 is

$$S_0 = 50 - 1.50e^{-0.1667\times0.1} = 48.52$$

As before $K = 50$, $r = 0.1$, $\sigma = 0.3$, and $T = 0.25$. In this case

$$d_1 = \frac{\ln(48.52/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.0414$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = -0.1086$$

The European put price is

$$50N(0.1086)e^{-0.1\times0.25} - 48.52N(-0.0414)$$

$$= 50 \times 0.5432e^{-0.1\times0.25} - 48.52 \times 0.4835 = 3.03$$

or \$3.03.

Problem 13.6.

What is implied volatility? How can it be calculated?

The implied volatility is the volatility that makes the Black–Scholes price of an option equal to its market price. It is calculated using an iterative procedure.

Problem 13.7.

A stock price is currently \$40. Assume that the expected return from the stock is 15% and that its volatility is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a two-year period?

In this case $\mu = 0.15$ and $\sigma = 0.25$. From equation (13.7) the probability distribution for the rate of return over a 2-year period with continuous compounding is:

$$\phi\left(0.15 - \frac{0.25^2}{2}, \frac{0.25^2}{2}\right)$$

i.e.,

$$\phi(0.11875, 0.03125)$$

The expected value of the return is 11.875% per annum and the standard deviation is $\sqrt{0.03125}$ or 17.68% per annum.

Problem 13.8.

A stock price follows geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is \$38.

(a) What is the probability that a European call option on the stock with an exercise price of \$40 and a maturity date in six months will be exercised?

(b) What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?

(a) The required probability is the probability of the stock price being above \$40 in six months' time. Suppose that the stock price in six months is S_T

$$\ln S_T \sim \phi(\ln 38 + (0.16 - \frac{0.35^2}{2})0.5,\ 0.35^2 \times 0.5)$$

i.e.,

$$\ln S_T \sim \phi(3.687, 0.06125)$$

Since $\ln 40 = 3.689$, the required probability is

$$1 - N\left(\frac{3.689 - 3.687}{\sqrt{0.06125}}\right) = 1 - N(0.008)$$

From normal distribution tables $N(0.008) = 0.5032$ so that the required probability is 0.4968. In general the required probability is $N(d_2)$. (See Problem 13.22).

(b) In this case the required probability is the probability of the stock price being less than \$40 in six months' time. It is

$$1 - 0.4968 = 0.5032$$

Problem 13.9.

Prove that with the notation in the chapter, a 95% confidence interval for S_T is between

$$S_0e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}} \quad \text{and} \quad S_0e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

From equation (13.3):

$$\ln S_T \sim \phi[\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T]$$

95% confidence intervals for $\ln S_T$ are therefore

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T - 1.96\sigma\sqrt{T}$$

and

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T + 1.96\sigma\sqrt{T}$$

95% confidence intervals for S_T are therefore

$$e^{\ln S_0+(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}}$$

and

$$e^{\ln S_0+(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

i.e.

$$S_0e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}}$$

and

$$S_0e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

Problem 13.10.

A portfolio manager announces that the average of the returns realized in each year of the last 10 years is 20% per annum. In what respect is this statement misleading?

The statement is misleading in that a certain sum of money, say $1000, when invested for 10 years in the fund would have realized a return (with annual compounding) of less than 20% per annum.

The average of the returns realized in each year is always greater than the return per annum (with annual compounding) realized over 10 years. The first is an arithmetic average of the returns in each year; the second is a geometric average of these returns.

Problem 13.11.

Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a security *that pays off a dollar amount equal to* $\ln S_T$ *at time* T *where* S_T *denotes the value of the stock price at time* T.

(a) *Use risk-neutral valuation to calculate the price of the security at time* t *in terms of the stock price,* S*, at time* t.

(b) *Confirm that your price satisfies the differential equation (13.16).*

(a) At time t, the expected value of $\ln S_T$ is, from equation (13.3)

$$\ln S + (\mu - \frac{\sigma^2}{2})(T - t)$$

In a risk-neutral world the expected value of $\ln S_T$ is therefore:

$$\ln S + (r - \frac{\sigma^2}{2})(T - t)$$

Using risk-neutral valuation the value of the security at time t is:

$$e^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T - t)\right]$$

(b) If:

$$f = e^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T - t)\right]$$

$$\frac{\partial f}{\partial t} = re^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T - t)\right] - e^{-r(T-t)}(r - \frac{\sigma^2}{2})$$

$$\frac{\partial f}{\partial S} = \frac{e^{-r(T-t)}}{S}$$

$$\frac{\partial^2 f}{\partial S^2} = -\frac{e^{-r(T-t)}}{S^2}$$

The left-hand side of the Black–Scholes–Merton differential equation is

$$e^{-r(T-t)}\left[r\ln S + r(r - \frac{\sigma^2}{2})(T - t) - (r - \frac{\sigma^2}{2}) + r - \frac{\sigma^2}{2}\right]$$

$$= re^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T - t)\right]$$

$$= rf$$

Hence equation (13.16) is satisfied.

Problem 13.12.

Consider a derivative that pays off S_T^n *at time* T *where* S_T *is the stock price at that time. When the stock price follows geometric Brownian motion, it can be shown that its price at time* t $(t \leq T)$ *has the form*

$$h(t, T)S^n$$

where S is the stock price at time t and h is a function only of t and T.

(a) By substituting into the Black–Scholes–Merton partial differential equation derive an ordinary differential equation satisfied by $h(t,T)$.

(b) What is the boundary condition for the differential equation for $h(t,T)$?

(c) Show that

$$h(t,T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$$

where r is the risk-free interest rate and σ is the stock price volatility.

This problem is related to Problem 12.10.

(a) If $G(S,t) = h(t,T)S^n$ then $\partial G/\partial t = h_t S^n$, $\partial G/\partial S = hnS^{n-1}$, and $\partial^2 G/\partial S^2 = hn(n-1)S^{n-2}$ where $h_t = \partial h/\partial t$. Substituting into the Black–Scholes–Merton differential equation we obtain

$$h_t + rhn + \frac{1}{2}\sigma^2 hn(n-1) = rh$$

(b) The derivative is worth S^n when $t = T$. The boundary condition for this differential equation is therefore $h(T,T) = 1$

(c) The equation

$$h(t,T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$$

satisfies the boundary condition since it collapses to $h = 1$ when $t = T$. It can also be shown that it satisfies the differential equation in (a). Alternatively we can solve the differential equation in (a) directly. The differential equation can be written

$$\frac{h_t}{h} = -r(n-1) - \frac{1}{2}\sigma^2 n(n-1)$$

The solution to this is

$$\ln h = [-r(n-1) - \frac{1}{2}\sigma^2 n(n-1)]t + k$$

where k is a constant. Since $\ln h = 0$ when $t = T$ it follows that

$$k = [r(n-1) + \frac{1}{2}\sigma^2 n(n-1)]T$$

so that

$$\ln h = [r(n-1) + \frac{1}{2}\sigma^2 n(n-1)](T-t)$$

or

$$h(t,T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$$

Problem 13.13.

What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is three months?

In this case $S_0 = 52$, $K = 50$, $r = 0.12$, $\sigma = 0.30$ and $T = 0.25$.

$$d_1 = \frac{\ln(52/50) + (0.12 + 0.3^2/2)0.25}{0.30\sqrt{0.25}} = 0.5365$$
$$d_2 = d_1 - 0.30\sqrt{0.25} = 0.3865$$

The price of the European call is

$$52N(0.5365) - 50e^{-0.12\times0.25}N(0.3865)$$
$$= 52 \times 0.7042 - 50e^{-0.03} \times 0.6504$$
$$= 5.06$$

or \$5.06.

Problem 13.14.

What is the price of a European put option on a non-dividend-paying stock when the stock price is \$69, the strike price is \$70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?

In this case $S_0 = 69$, $K = 70$, $r = 0.05$, $\sigma = 0.35$ and $T = 0.5$.

$$d_1 = \frac{\ln(69/70) + (0.05 + 0.35^2/2) \times 0.5}{0.35\sqrt{0.5}} = 0.1666$$
$$d_2 = d_1 - 0.35\sqrt{0.5} = -0.0809$$

The price of the European put is

$$70e^{-0.05\times0.5}N(0.0809) - 69N(-0.1666)$$
$$= 70e^{-0.025} \times 0.5323 - 69 \times 0.4338$$
$$= 6.40$$

or \$6.40.

Problem 13.15.

Consider an American call option on a stock. The stock price is \$70, the time to maturity is eight months, the risk-free rate of interest is 10% per annum, the exercise price is \$65, and the volatility is 32%. A dividend of \$1 is expected after three months and again after six months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Use DerivaGem to calculate the price of the option.

Using the notation of Section 13.12, $D_1 = D_2 = 1$, $K(1 - e^{-r(T-t_2)}) = 65(1 - e^{-0.1\times0.1667}) = 1.07$, and $K(1 - e^{-r(t_2-t_1)}) = 65(1 - e^{-0.1\times0.25}) = 1.60$. Since

$$D_1 < K(1 - e^{-r(T-t_2)})$$

and

$$D_2 < K(1 - e^{-r(t_2 - t_1)})$$

It is never optimal to exercise the call option early. DerivaGem shows that the value of the option is 10.94.

Problem 13.16.

A call option on a non-dividend-paying stock has a market price of \$$2\frac{1}{2}$. The stock price is \$15, the exercise price is \$13, the time to maturity is three months, and the risk-free interest rate is 5% per annum. What is the implied volatility?

In the case $c = 2.5$, $S_0 = 15$, $K = 13$, $T = 0.25$, $r = 0.05$. The implied volatility must be calculated using an iterative procedure.

A volatility of 0.2 (or 20% per annum) gives $c = 2.20$. A volatility of 0.3 gives $c = 2.32$. A volatility of 0.4 gives $c = 2.507$. A volatility of 0.39 gives $c = 2.487$. By interpolation the implied volatility is about 0.397 or 39.7% per annum.

Problem 13.17.

With the notation used in this chapter

(a) What is $N'(x)$?

(b) Show that $SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$, where S is the stock price at time t

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

(c) Calculate $\partial d_1/\partial S$ and $\partial d_2/\partial S$.

(d) Show that when

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T - t}}$$

where c is the price of a call option on a non-dividend-paying stock.

(e) Show that $\partial c/\partial S = N(d_1)$.

(f) Show that the c satisfies the Black–Scholes–Merton differential equation.

(g) Show that c satisfies the boundary condition for a European call option, i.e., that $c = \max(S - K, 0)$ as $t \longrightarrow T$

(a) Since $N(x)$ is the cumulative probability that a variable with a standardized normal distribution will be less than x, $N'(x)$ is the probability density function for a standardized normal distribution, that is,

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

(b)

$$N'(d_1) = N'(d_2 + \sigma\sqrt{T-t})$$
$$= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{d_2^2}{2} - \sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right]$$
$$= N'(d_2) \exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right]$$

Because

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

it follows that

$$\exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right] = \frac{Ke^{-r(T-t)}}{S}$$

As a result

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

which is the required result.

(c)

$$d_1 = \frac{\ln\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$
$$= \frac{\ln S - \ln K + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Hence

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Similarly

$$d_2 = \frac{\ln S - \ln K + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$\frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Therefore:

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$$

(d)

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$\frac{\partial c}{\partial t} = SN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}$$

From (b):

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) + SN'(d_1)\left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right)$$

Since

$$\begin{aligned}
d_1 - d_2 &= \sigma\sqrt{T-t} \\
\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} &= \frac{\partial}{\partial t}(\sigma\sqrt{T-t}) \\
&= -\frac{\sigma}{2\sqrt{T-t}}
\end{aligned}$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

(e) From differentiating the Black–Scholes–Merton formula for a call price we obtain

$$\frac{\partial c}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{dS}$$

From the results in (b) and (c) it follows that

$$\frac{\partial c}{\partial S} = N(d_1)$$

(f) Differentiating the result in (e) and using the result in (c), we obtain

$$\begin{aligned}
\frac{\partial^2 c}{\partial S^2} &= N'(d_1)\frac{\partial d_1}{\partial S} \\
&= N'(d_1)\frac{1}{S\sigma\sqrt{T-t}}
\end{aligned}$$

From the results in d) and e)

$$\begin{aligned}
\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S^2} &= -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} \\
&+ rSN(d_1) + \frac{1}{2}\sigma^2 S^2 N'(d_1)\frac{1}{S\sigma\sqrt{T-t}} \\
&= r[SN(d_1) - Ke^{-r(T-t)}N(d_2)] \\
&= rc
\end{aligned}$$

This shows that the Black–Scholes formula for a call option does indeed satisfy the Black–Scholes–Merton differential equation

(g) Consider what happens in the formula for c in part (d) as t approaches T. If $S > K$, d_1 and d_2 tend to infinity and $N(d_1)$ and $N(d_2)$ tend to 1. If $S < K$, d_1 and d_2 tend to zero. It follows that the formula for c tends to $\max(S - K, 0)$.

Problem 13.18.

Show that the Black–Scholes formulas for call and put options satisfy put–call parity.

From the Black–Scholes equations

$$p + S_0 = Ke^{-rT}N(-d_2) - S_0N(-d_1) + S_0$$

Because $1 - N(-d_1) = N(d_1)$ this is

$$Ke^{-rT}N(-d_2) + S_0N(d_1)$$

Also:

$$c + Ke^{-rT} = S_0N(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT}$$

Because $1 - N(d_2) = N(-d_2)$, this is also

$$Ke^{-rT}N(-d_2) + S_0N(d_1)$$

The Black–Scholes equations are therefore consistent with put–call parity.

Problem 13.19.

A stock price is currently \$50 and the risk-free interest rate is 5%. Use the DerivaGem software to translate the following table of European call options on the stock into a table of implied volatilities, assuming no dividends. Are the option prices consistent with the assumptions underlying Black–Scholes?

	Maturity (months)		
Strike Price (\$)	*3*	*6*	*12*
45	*7.0*	*8.3*	*10.5*
50	*3.7*	*5.2*	*7.5*
55	*1.6*	*2.9*	*5.1*

This problem naturally leads on to the material in Chapter 18 on volatility smiles. Using DerivaGem we obtain the following table of implied volatilities:

	Maturity (months)		
Strike Price ($)	3	6	12
45	37.78	34.99	34.02
50	34.15	32.78	32.03
55	31.98	30.77	30.45

The option prices are not exactly consistent with Black–Scholes. If they were, the implied volatilities would be all the same. We usually find in practice that low strike price options on a stock have significantly higher implied volatilities than high strike price options on the same stock.

Problem 13.20.

Explain carefully why Black's approach to evaluating an American call option on a dividend-paying stock may give an approximate answer even when only one dividend is anticipated. Does the answer given by Black's approach understate or overstate the true option value? Explain your answer.

Black's approach in effect assumes that the holder of option must decide at time zero whether it is a European option maturing at time t_n (the final ex-dividend date) or a European option maturing at time T. In fact the holder of the option has more flexibility than this. The holder can choose to exercise at time t_n if the stock price at that time is above some level but not otherwise. Furthermore, if the option is not exercised at time t_n, it can still be exercised at time T.

It appears from this argument that Black's approach understates the true option value. However, the way in which volatility is applied can lead to Black's approach overstating the option value. Black applies the volatility to the option price. The binomial model, as we will see in Chapter 19, applies the volatility to the stock price less the present value of the dividend. This issue is also discussed in Example 13.10.

Problem 13.21.

Consider an American call option on a stock. The stock price is $50, the time to maturity is 15 months, the risk-free rate of interest is 8% per annum, the exercise price is $55, and the volatility is 25%. Dividends of $1.50 are expected in 4 months and 10 months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Calculate the price of the option.

With the notation in the text

$$D_1 = D_2 = 1.50, \quad t_1 = 0.3333, \quad t_2 = 0.8333, \quad T = 1.25, \quad r = 0.08 \quad \text{and} \quad K = 55$$

$$K\left[1 - e^{-r(T-t_2)}\right] = 55(1 - e^{-0.08\times 0.4167}) = 1.80$$

Hence

$$D_2 < K\left[1 - e^{-r(T-t_2)}\right]$$

Also:

$$K\left[1-e^{-r(t_2-t_1)}\right] = 55(1-e^{-0.08\times 0.5}) = 2.16$$

Hence:

$$D_1 < K\left[1-e^{-r(t_2-t_1)}\right]$$

It follows from the conditions established in Section 13.12 that the option should never be exercised early.

The present value of the dividends is

$$1.5e^{-0.3333\times 0.08} + 1.5e^{-0.8333\times 0.08} = 2.864$$

The option can be valued using the European pricing formula with:

$$S_0 = 50 - 2.864 = 47.136, \quad K = 55, \quad \sigma = 0.25, \quad r = 0.08, \quad T = 1.25$$

$$d_1 = \frac{\ln(47.136/55) + (0.08 + 0.25^2/2)1.25}{0.25\sqrt{1.25}} = -0.0545$$

$$d_2 = d_1 - 0.25\sqrt{1.25} = -0.3340$$

$$N(d_1) = 0.4783, \quad N(d_2) = 0.3692$$

and the call price is

$$47.136 \times 0.4783 - 55e^{-0.08\times 1.25} \times 0.3692 = 4.17$$

or \$4.17.

Problem 13.22.

Show that the probability that a European call option will be exercised in a risk-neutral world is, with the notation introduced in this chapter, $N(d_2)$. What is an expression for the value of a derivative that pays off \$100 if the price of a stock at time T is greater than K?

The probability that the call option will be exercised is the probability that $S_T > K$ where S_T is the stock price at time T. In a risk neutral world

$$\ln S_T \sim \phi[\ln S_0 + (r - \sigma^2/2)T, \sigma^2 T]$$

The probability that $S_T > K$ is the same as the probability that $\ln S_T > \ln K$. This is

$$1 - N\left[\frac{\ln K - \ln S_0 - (r-\sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

$$= N\left[\frac{\ln(S_0/K) + (r-\sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

$$= N(d_2)$$

The expected value at time T in a risk neutral world of a derivative security which pays off \$100 when $S_T > K$ is therefore

$$100N(d_2)$$

From risk neutral valuation the value of the security at time t is

$$100e^{-rT}N(d_2)$$

Problem 13.23.

Show that S^{-2r/σ^2} could be the price of a traded security.

If $f = S^{-2r/\sigma^2}$ then

$$\frac{\partial f}{\partial S} = -\frac{2r}{\sigma^2}S^{-2r/\sigma^2-1}$$

$$\frac{\partial^2 f}{\partial S^2} = \left(\frac{2r}{\sigma^2}\right)\left(\frac{2r}{\sigma^2}+1\right)S^{-2r/\sigma^2-2}$$

$$\frac{\partial f}{\partial t} = 0$$

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rS^{-2r/\sigma^2} = rf$$

This shows that the Black–Scholes equation is satisfied. S^{-2r/σ^2} could therefore be the price of a traded security.

Problem 13.24.

A company has an issue of executive stock options outstanding. Should dilution be taken into account when the options are valued? Explain you answer.

The answer is no. If markets are efficient they have already taken potential dilution into account in determining the stock price. This argument is explained in Business Snapshot 13.3.

Problem 13.25.

A company's stock price is \$50 and 10 million shares are outstanding. The company is considering giving its employees three million at-the-money five-year call options. Option exercises will be handled by issuing more shares. The stock price volatility is 25%, the five-year risk-free rate is 5% and the company does not pay dividends. Estimate the cost to the company of the employee stock option issue.

The Black-Scholes price of the option is given by setting $S_0 = 50$, $K = 50$, $r = 0.05$, $\sigma = 0.25$, and $T = 5$. It is 16.252. From an analysis similar to that in Section 13.10 the cost to the company of the options is

$$\frac{10}{10+3} \times 16.252 = 12.5$$

or about \$12.5 per option. The total cost is therefore 3 million times this or \$37.5 million. If the market perceives no benefits from the options the stock price will fall by \$3.75.

ASSIGNMENT QUESTIONS

Problem 13.26.

A stock price is currently \$50. Assume that the expected return from the stock is 18% and its volatility is 30%. What is the probability distribution for the stock price in two years? Calculate the mean and standard deviation of the distribution. Determine the 95% confidence interval.

In this case $S_0 = 50$, $\mu = 0.18$ and $\sigma = 0.30$. The probability distribution of the stock price in two years, S_T, is lognormal and is, from equation (13.3), given by:

$$\ln S_T \sim \phi[\ln 50 + (0.18 - \frac{0.09}{2})2,\ 0.3^2 \times 2]$$

i.e.,

$$\ln S_T \sim \phi(4.18,\ 0.18)$$

The mean stock price is from equation (13.4)

$$50e^{2\times 0.18} = 50e^{0.36} = 71.67$$

and the standard deviation is, from equation (13.5),

$$50e^{2\times 0.18}\sqrt{e^{0.09\times 2} - 1} = 31.83$$

95% confidence intervals for $\ln S_T$ are

$$4.18 - 1.96 \times 0.42 \quad \text{and} \quad 4.18 + 1.96 \times 0.42$$

i.e.,

$$3.35 \quad \text{and} \quad 5.01$$

These correspond to 95% confidence limits for S_T of

$$e^{3.35} \quad \text{and} \quad e^{5.01}$$

i.e.,

$$28.52 \quad \text{and} \quad 150.44$$

Problem 13.27.

Suppose that observations on a stock price (in dollars) at the end of each of 15 consecutive weeks are as follows:

$$30.2, 32.0, 31.1, 30.1, 30.2, 30.3, 30.6, 33.0, 32.9, 33.0, 33.5, 33.5, 33.7, 33.5, 33.2$$

Estimate the stock price volatility. What is the standard error of your estimate?

The calculations are shown in the table below

$$\sum u_i = 0.09471 \quad \sum u_i^2 = 0.01145$$

and an estimate of standard deviation of weekly returns is:

$$\sqrt{\frac{0.01145}{13} - \frac{0.09471^2}{14 \times 13}} = 0.02884$$

The volatility per annum is therefore $0.02884\sqrt{52} = 0.2079$ or 20.79%. The standard error of this estimate is

$$\frac{0.2079}{\sqrt{2 \times 14}} = 0.0393$$

or 3.9% per annum.

Problem 13.27
Computation of Volatility

Week	Closing Stock Price ($)	Price Relative $= S_i/S_{i-1}$	Daily Return $u_i = \ln(S_i/S_{i-1})$
1	30.2		
2	32.0	1.05960	0.05789
3	31.1	0.97188	-0.02853
4	30.1	0.96785	-0.03268
5	30.2	1.00332	0.00332
6	30.3	1.00331	0.00331
7	30.6	1.00990	0.00985
8	33.0	1.07843	0.07551
9	32.9	0.99697	-0.00303
10	33.0	1.00304	0.00303
11	33.5	1.01515	0.01504
12	33.5	1.00000	0.00000
13	33.7	1.00597	0.00595
14	33.5	0.99407	-0.00595
15	33.2	0.99104	-0.00900

Problem 13.28.
A financial institution plans to offer a security that pays off a dollar amount equal to S_T^2 at time T.

(a) *Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S, at time t. (Hint: The expected value of S_T^2 can be calculated from the mean and variance of S_T given in section 13.1.)*

(b) *Confirm that your price satisfies the differential equation (13.16).*

(a) The expected value of the security is $E[(S_T)^2]$ From equations (13.4) and (13.5), at time t :

$$E(S_T) = Se^{\mu(T-t)}$$

$$\text{var}(S_T) = S^2 e^{2\mu(T-t)}[e^{\sigma^2(T-t)} - 1]$$

Since $\text{var}(S_T) = E[(S_T)^2] - [E(S_T)]^2$, it follows that $E[(S_T)^2] = \text{var}(S_T) + [E(S_T)]^2$ so that

$$E[(S_T)^2] = S^2 e^{2\mu(T-t)}[e^{\sigma^2(T-t)} - 1] + S^2 e^{2\mu(T-t)}$$
$$= S^2 e^{(2\mu+\sigma^2)(T-t)}$$

In a risk-neutral world $\mu = r$ so that

$$\hat{E}[(S_T)^2] = S^2 e^{(2r+\sigma^2)(T-t)}$$

Using risk-neutral valuation, the value of the derivative security at time t is

$$e^{-r(T-t)}\hat{E}[(S_T)^2]$$

$$= S^2 e^{(2r+\sigma^2)(T-t)} e^{-r(T-t)}$$
$$= S^2 e^{(r+\sigma^2)(T-t)}$$

(b) If:

$$f = S^2 e^{(r+\sigma^2)(T-t)}$$
$$\frac{\partial f}{\partial t} = -S^2(r+\sigma^2)e^{(r+\sigma^2)(T-t)}$$
$$\frac{\partial f}{\partial S} = 2Se^{(r+\sigma^2)(T-t)}$$
$$\frac{\partial^2 f}{\partial S^2} = 2e^{(r+\sigma^2)(T-t)}$$

The left-hand side of the Black-Scholes–Merton differential equation is:

$$-S^2(r+\sigma^2)e^{(r+\sigma^2)(T-t)} + 2rS^2 e^{(r+\sigma^2)(T-t)} + \sigma^2 S^2 e^{(r+\sigma^2)(T-t)}$$
$$= rS^2 e^{(r+\sigma^2)(T-t)}$$
$$= rf$$

Hence the Black-Scholes equation is satisfied.

Problem 13.29.

Consider an option on a non-dividend-paying stock when the stock price is \$30, the exercise price is \$29, the risk-free interest rate is 5%, the volatility is 25% per annum, and the time to maturity is 4 months.

(a) What is the price of the option if it is a European call?
(b) What is the price of the option if it is an American call?
(c) What is the price of the option if it is a European put?
(d) Verify that put–call parity holds.

In this case $S_0 = 30$, $K = 29$, $r = 0.05$, $\sigma = 0.25$ and $T = 0.3333$

$$d_1 = \frac{\ln(30/29) + (0.05 + 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.4225$$

$$d_2 = \frac{\ln(30/29) + (0.05 - 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.2782$$

$$N(0.4225) = 0.6637, \quad N(0.2782) = 0.6096$$

$$N(-0.4225) = 0.3363, \quad N(-0.2782) = 0.3904$$

(a) The European call price is

$$30 \times 0.6637 - 29e^{-0.05\times0.3333} \times 0.6096 = 2.52$$

or $2.52.

(b) The American call price is the same as the European call price. It is $2.52.

(c) The European put price is

$$29e^{-0.05\times0.3333} \times 0.3904 - 30 \times 0.3363 = 1.05$$

or $1.05.

(d) Put-call parity states that:

$$p + S_0 = c + Ke^{-rT}$$

In this case $c = 2.52$, $S_0 = 30$, $K = 29$, $p = 1.05$ and $e^{-rT} = 0.9835$ and it is easy to verify that the relationship is satisfied.

Problem 13.30.

Assume that the stock in Problem 13.29 is due to go ex-dividend in $1\frac{1}{2}$ months. The expected dividend is 50 cents.

(a) What is the price of the option if it is a European call?

(b) What is the price of the option if it is a European put?

(c) If the option is an American call, are there any circumstances under which it will be exercised early?

(a) The present value of the dividend must be subtracted from the stock price. This gives a new stock price of:

$$30 - 0.5e^{-0.125\times0.05} = 29.5031$$

and

$$d_1 = \frac{\ln(29.5031/29) + (0.05 + 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.3068$$

$$d_2 = \frac{\ln(29.5031/29) + (0.05 - 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.1625$$

$$N(d_1) = 0.6205; \quad N(d_2) = 0.5645$$

The price of the option is therefore

$$29.5031 \times 0.6205 - 29e^{-0.3333\times0.05} \times 0.5645 = 2.21$$

or $2.21.

(b) Since

$$N(-d_1) = 0.3795, \quad N(-d_2) = 0.4355$$

the value of the option when it is a European put is

$$29e^{-0.3333\times0.05} \times 0.4355 - 29.5031 \times 0.3795 = 1.22$$

or \$1.22.

(c) If t_1 denotes the time when the dividend is paid:

$$K[1 - e^{-r(T-t_1)}] = 29(1 - e^{-0.05\times0.2083}) = 0.3005$$

This is less than the dividend. Hence the option should be exercised immediately before the ex-dividend date for a sufficiently high value of the stock price.

Problem 13.31.

Consider an American call option when the stock price is \$18, the exercise price is \$20, the time to maturity is six months, the volatility is 30% per annum, and the risk-free interest rate is 10% per annum. Two equal dividends are expected during the life of the option with ex-dividend dates at the end of two months and five months. Assume the dividends are 40 cents. Use Black's approximation and the DerivaGem software to value the option. How high can the dividends be without the American option being worth more than the corresponding European option?

We first value the option assuming that it is not exercised early, we set the time to maturity equal to 0.5. There is a dividend of 0.4 in 2 months and 5 months. Other parameters are $S_0 = 18$, $K = 20$, $r = 10\%$, $\sigma = 30\%$. DerivaGem gives the price as 0.7947. We next value the option assuming that it is exercised at the five-month point just before the final dividend. DerivaGem gives the price as 0.7668. The price given by Black's approximation is therefore 0.7947. DerivaGem also shows that the correct American option price calculated with 100 time steps is 0.8243.

It is never optimal to exercise the option immediately before the first ex-dividend date when

$$D_1 \leq K[1 - e^{-r(t_2-t_1)}]$$

where D_1 is the size of the first dividend, and t_1 and t_2 are the times of the first and second dividend respectively. Hence we must have:

$$D_1 \leq 20[1 - e^{-(0.1\times0.25)}]$$

that is,

$$D_1 \leq 0.494$$

It is never optimal to exercise the option immediately before the second ex-dividend date when:

$$D_2 \leq K(1 - e^{-r(T-t_2)})$$

where D_2 is the size of the second dividend. Hence we must have:

$$D_2 \leq 20(1 - e^{-0.1\times0.0833})$$

that is,

$$D_2 \leq 0.166$$

It follows that the dividend can be as high as 16.6 cents per share without the American option being worth more than the corresponding European option.

CHAPTER 14
Employee Stock Options

Notes for the Instructor

This chapter is new to the seventh edition. Employee stock options have been much in the news in recent years and I find students enjoy talking about them. Many students hope to become rich one day by exercising such options!

The chapter covers how the options typically work, whether they align the interests of senior executives and shareholders, their accounting treatment, alternative valuation approaches, and backdating scandals. Many instructors will want to spend time on the academic research of Yermack, Lie, and Heron which was largely responsible for exposing the backdating scandals (see Section 14.5). Others may want to focus on how employee stock options can be designed to better align the interests of shareholders and senior managers. As described in Section 14.1 the traditional stock option plan is one where at-the-money options are issued periodically. For many years, companies were reluctant to move away from this type of plan because they would then be required to expense the options. Now the accounting treatment of employee stock options has now changed (with expensing being mandatory) and so there is no reason for companies not to consider nontraditional plans such as those mentioned in Section 14.3.

I recommend spending some time talking about the fact that employee stock options (unlike regular call options) cannot be sold. This leads to the situation where they tend to be exercised much earlier than regular call options (see Section 14.1). A discussion of this should reinforce a student's understanding of the arguments concerning early exercise of calls in Chapter 9.

The three assignment questions test whether students can use some of the approaches for valuing employee stock options. If 14.14 is assigned it is a good idea to suggest to students that they calculated the expected life using a tree.

QUESTIONS AND PROBLEMS

Problem 14.1.

Why was it attractive for companies to grant at-the-money stock options prior to 2005? What changed in 2005?

Prior to 2005 companies did not have to expense at-the-money options on the income statement. They merely had to report the value of the options in notes to the accounts. FAS 123 and IAS 2 required the fair value of the options to be reported as a cost on the income statement starting in 2005.

Problem 14.2.

What are the main differences between a typical employee stock option and a call option traded on an exchange or in the over-the-counter market?

The main differences are a) employee stock options last much longer than the typical exchange-traded or over-the-counter option, b) there is usually a vesting period during which they cannot be exercised, c) the options cannot be sold by the employee, d) if the employee leaves the company the options usually either expire worthless or have to be exercised immediately, and e) exercise of the options usually leads to the company issuing more shares.

Problem 14.3.

Explain why employee stock options on a non-dividend-paying stock are frequently exercised before the end of their lives whereas an exchange-traded call option on such a stock is never exercised early.

It is always better for the option holder to sell a call option on a non-dividend -paying stock rather than exercise it. Employee stock options cannot be sold and so the only way an employee can monetize the option is to exercise the option and sell the stock.

Problem 14.4.

"Stock option grants are good because they motivate executives to act in the best interests of shareholders." Discuss this viewpoint.

This is questionable. Executives benefit from share price increases but do not bear the costs of share price decreases. Employee stock options are liable to encourage executives to take decisions that boost the value of the stock in the short term at the expense of the long term health of the company. It may even be the case that executives are encouraged to take high risks so as to maximize the value of their options.

Problem 14.5.

"Granting stock options to executives is like allowing a professional footballer to bet on the outcome of games." Discuss this viewpoint.

Professional footballers are not allowed to bet on the outcomes of games because they themselves influence the outcomes. Arguably, an executive should not be allowed to bet on the future stock price of her company because her actions influence that price. However, it could be argued that there is nothing wrong with a professional footballer betting that his team will win (but everything wrong with betting that it will lose). Similarly there is nothing wrong with an executive betting that her company will do well.

Problem 14.6.

Why did some companies backdate stock option grants in the US prior to 2002? What changed in 2002?

Backdating allowed the company to issue employee stock options with a strike price equal to the price at some previous date and claim that they were at the money. At the money options did not lead to an expense on the income statement until 2005. The amount recorded for the value of the options in the notes to the income was less than the actual cost on the true grant date. In 2002 the SEC required companies to report stock option grants within two business days of the grant date. This eliminated the possibility of backdating for companies that complied with this rule.

Problem 14.7.

In what way would the benefits of backdating be reduced if a stock option grant had to be revalued at the end of each quarter?

If a stock option grant had to be revalued each quarter the value of the option of the grant date (true or fabricated) would become less important. Stock price movements following the reported grant date would be incorporated in the next revaluation. The total cost of the options would be independent of the stock price on the grant date.

Problem 14.8.

Explain how you would do the analysis to produce a chart such as the one in Figure 15.2.

It would be necessary to look at returns on each stock in the sample (possibly adjusted for the returns on the market and the beta of the stock) around the reported employee stock option grant date. One could designate Day 0 as the grant date and look at returns on each stock each day from Day –30 to Day +30. The returns would then be averaged across the stocks.

Problem 14.9.

On May 31 a company's stock price is \$70. One million shares are outstanding. An executive exercises 100,000 stock options with a strike price of \$50. What is the impact of this on the stock price?

There should be no impact on the stock price because the stock price will already reflect the dilution expected from the executive's exercise decision.

Problem 14.10.

The notes accompanying a company's financial statements say: "Our executive stock options last 10 years and vest after four years. We valued the options granted this year using the Black–Scholes model with an expected life of 5 years and a volatility of 20%. "What does this mean? Discuss the modeling approach used by the company.

The notes in indicate that the Black–Scholes model was used to produce the valuation with T the option life being set equal to5 years and the stock price volatility being set equal to 20%.

Problem 14.11.

In a Dutch auction of 10,000 options, bids are as follows
A bids \$30 for 3,000
B bids \$33 for 2,500

C bids \$29 for 5,000
D bids \$40 for 1,000
E bids \$22 for 8,000
F bids \$35 for 6,000
What is the result of the auction? Who buys how many at what price?

The price at which 10,000 options can be sold is \$30. B, D, and F get their order completely filled at this price. A buys 500 options (out of its total bid for 3,000 options) at this price.

Problem 14.12.

A company has granted 500,000 options to its executives. The stock price and strike price are both \$40. The options last for 12 years and vest after four years. The company decides to value the options using an expected life of five years and a volatility of 30% per annum. The company pays no dividends and the risk-free rate is 4%. What will the company report as an expense for the options on its income statement?

The options are valued using Black–Scholes with $S_0 = 40$, $K = 40$, $T = 5$, $\sigma = 0.3$ and $r = 0.04$. The value of each option is \$4.488. The total expense reported is $500,000 \times \$4.488$ or \$2.244 million.

Problem 14.13.

A company's CFO says: "The accounting treatment of stock options is crazy. We granted 10,000,000 at-the-money stock options to our employees last year when the stock price was \$30. We estimated the value of each option on the grant date to be \$5. At our year end the stock price had fallen to \$4, but we were still stuck with a \$50 million charge to the P&L." Discuss.

The problem is that under the current rules the options are valued only once—on the grant date. Arguably it would make sense to treat the options in the same way as other derivatives entered into by the company and revalue them on each reporting date. However, this does not happen under the current rules in the United States unless the options are settled in cash.

ASSIGNMENT QUESTIONS

Problem 14.14.

What is the (risk-neutral) expected life for the employee stock option in Example 14.2? What is the value of the option obtained by using this expected life in Black–Scholes?

The expected life at time zero can be calculated by rolling back through the tree asking the question at each node: "What is the expected life if the node is reached." This is what has been done in Figure M14.1. For example at node G (time 6 years) there is a 81% chance that the option will be exercised and a 19% chance that it will last an extra two years. The expected life if node G is reached is therefore $0.81 \times 6 + 0.19 \times 8 = 6.38$

years. Similarly, the expected life if node H is reached is $0.335 \times 6 + 0.665 \times 8 = 7.33$ years. The expected life if node I or J is reached is $0.05 \times 6 + 0.95 \times 8 = 7.90$ years. The expected life if node D is reached is

$$0.43 \times 4 + 0.57 \times (0.5158 \times 6.38 + 0.4842 \times 7.33) = 5.62$$

Continuing in this way the expected life at time zero is 6.86 years. (As in Example 14.2 we assume that no employees leave at time zero.)

The value of the option assuming an expected life of 6.86 years is given by Black–Scholes with $S_0 = 40$, $K = 40$, $r = 0.05$, $\sigma = 0.3$ and $T = 6.86$. It is 17.17. Using a four-step tree it is 16.51.

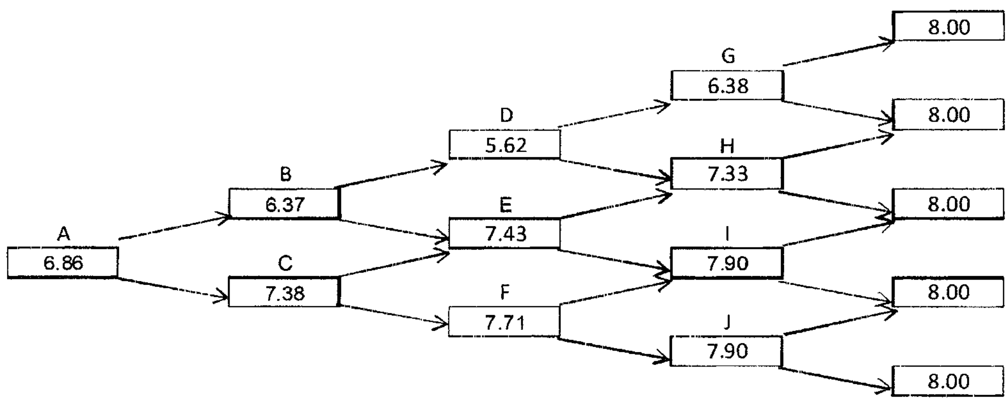


Figure M14.1 Tree for calculating expected life in Problem 14.14

Problem 14.15.

A company has granted 2,000,000 options to its employees. The stock price and strike price are both \$60. The options last for 8 years and vest after two years. The company decides to value the options using an expected life of six years and a volatility of 22% per annum. The dividend on the stock is \$1, payable half way through each year, and the risk-free rate is 5%. What will the company report as an expense for the options on its income statement.

The options are valued using Black–Scholes with $K = 60$, $T = 6$, $\sigma = 0.22$, $r = 0.05$. The present value of the dividends during the six years assumed life are

$$1 \times e^{-0.05\times 0.5} + 1 \times e^{-0.05\times 1.5} + 1 \times e^{-0.05\times 2.5} + 1 \times e^{-0.05\times 3.5} + 1 \times e^{-0.05\times 4.5} + 1 \times e^{-0.05\times 5.5}$$

$$= 5.183$$

The stock price, S_0, adjusted for dividend is therefore $60 - 5.183 = 54.817$. The Black–Scholes model gives the price of one option as \$16.492. The company will therefore report as an expense $2,000,000 \times \$16.492$ or \$32.984 million.

Problem 14.16.

A company has granted 1,000,000 options to its employees. The stock price and strike price are both \$20. The options last 10 years and vest after three years. The stock price volatility is 30%, the risk-free rate is 5%, and the company pays no dividends. Use a four-step tree to value the options. Assume that there is a probability of 4% that an employee leaves the company at the beginning of each the time steps on your tree. Assume also that the probability of voluntary early exercise at a node, conditional on no prior exercise, when a) the option has vested and b) the option is in the money, is

$$1 - \exp[-a(S/K - 1)/T]$$

where S is the stock price, K is the strike price, T is the time to maturity and $a = 2$.

The valuation is shown in Figure M14.2. The tree is similar to Figure 14.1 in the text. The upper number at each node is the stock price and the lower number is the value of the option. In this case $u = 1.6070$ and $p = 0.5188$. The probability of voluntary exercise at nodes A, B, and C are 0.4690, 0.9195, and 0.3846, respectively. The total probability of exercise at these nodes (including the impact of employees leaving the company) is 0.4902, 0.9227, and 0.4093. The value of each option is \$8.54 and the value of the option grant is \$8.54 million. This problem and Example 14.2 in the text specify that employees are assumed to leave at the beginning of each time period. It is questionable whether this includes time zero. Both my answer to this question and the answer to Example 14.1 assume that it does not include time zero. (On reflection, it would have been better for both questions to say that employees leave at the end of each time period.) If in this question it is assumed that 4% of employees leave the company at the initial node the answer is reduced by 4% to \$8.20 million.

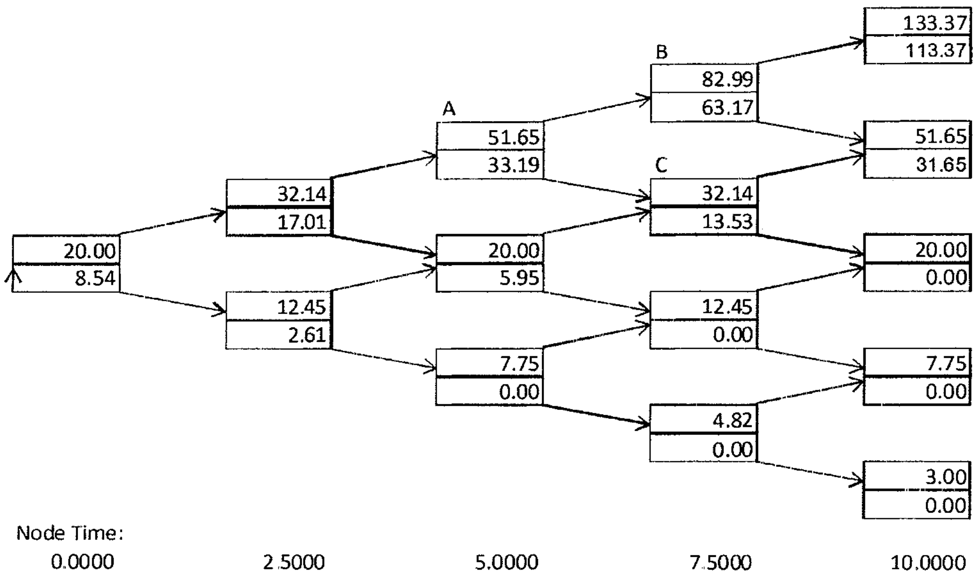


Figure M14.2 Valuation of employee stock option in Problem 14.16

CHAPTER 15
Options on Stock Indices and Currencies

Notes for the Instructor

The chapter concerned with options on stock indices, currencies, and futures in the sixth edition has been split into two chapters (15 and 16) in the seventh edition. Chapter 15 is concerned with options on stock indices and currencies; Chapter 16 is concerned with options on futures.

The material on options on stock indices and currencies has also been restructured for the seventh edition to make it more interesting. Instead of starting with valuation, it now starts with examples of how options on stock indices and options on foreign currencies are used. Range forwards are discussed here rather than later in the book.

For students who have a good knowledge of Chapter 13, the valuation material in this chapter should present few problems. The key argument is in Section 15.3 and shows how the Black-Scholes formulas can be modified to provide valuations of European call and put options on a stock paying a known dividend yield. Stock indices and currencies are analogous to stocks paying known dividend yields.

Any of Problems 15.23 to 13.28 make good assignment questions. Problem 15.22 requires Ito's lemma to have been covered.

QUESTIONS AND PROBLEMS

Problem 15.1.

A portfolio is currently worth $10 million and has a beta of 1.0. An index is currently standing at 800. Explain how a put option on the index with a strike of 700 can be used to provide portfolio insurance.

When the index goes down to 700, the value of the portfolio can be expected to be $10 \times (700/800) =$ $8.75 million. (This assumes that the dividend yield on the portfolio equals the dividend yield on the index.) Buying put options on 10,000,000/800 = 12,500 times the index with a strike of 700 therefore provides protection against a drop in the value of the portfolio below $8.75 million. If each contract is on 100 times the index a total of 125 contracts would be required.

Problem 15.2.

"Once we know how to value options on a stock paying a dividend yield, we know how to value options on stock indices, currencies, and futures." Explain this statement.

A stock index is analogous to a stock paying a continuous dividend yield, the dividend yield being the dividend yield on the index. A currency is analogous to a stock paying a continuous dividend yield, the dividend yield being the foreign risk-free interest rate.

Problem 15.3.

A stock index is currently 300, the dividend yield on the index is 3% per annum, and the risk-free interest rate is 8% per annum. What is a lower bound for the price of a six-month European call option on the index when the strike price is 290?

The lower bound is given by equation 15.1 as

$$300e^{-0.03\times0.5} - 290e^{-0.08\times0.5} = 16.90$$

Problem 15.4.

A currency is currently worth \$0.80 and has a volatility of 12%. The domestic and foreign risk-free interest rates are 6% and 8%, respectively. Use a two-step binomial tree to value a) a European four-month call option with a strike price of \$0.79 and b) an American four-month call option with the same strike price

In this case $u = 1.0502$ and $p = 0.4538$. The tree is shown in Figure S15.1. The value of the option if it is European is \$0.0235. the value of the option if it is American is \$0.0250.

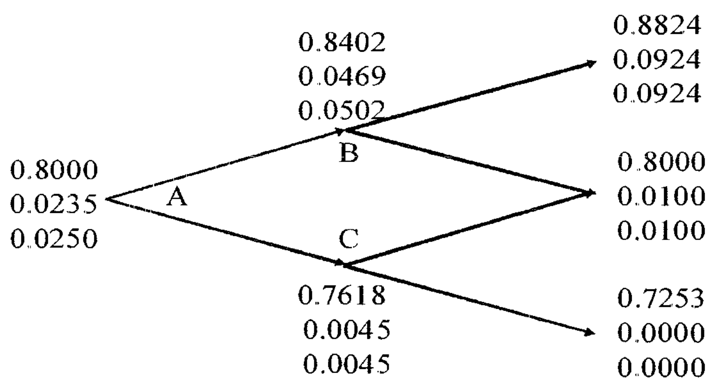


Figure S15.1 Tree to evaluate European and American put options in Problem 15.4. At each node, upper number is the stock price; next number is the European put price; final number is the American put price

Problem 15.5.

Explain how corporations use range forward contracts to hedge their foreign exchange risk.

A range forward contract allows a corporation to ensure that the exchange rate applicable to a transaction will not be worse that one exchange rate and will not be better than another exchange rate. Depending on the exposure being hedged a range forward contract is created by either a) buying a put with the lower exchange rate and selling a call with the higher exchange rate or b) selling a put with the lower exchange rate and buying a call with the higher exchange rate.

Problem 15.6.

Calculate the value of a three-month at-the-money European call option on a stock index when the index is at 250, the risk-free interest rate is 10% per annum, the volatility of the index is 18% per annum, and the dividend yield on the index is 3% per annum.

In this case, $S_0 = 250$, $K = 250$, $r = 0.10$, $\sigma = 0.18$, $T = 0.25$, $q = 0.03$ and

$$d_1 = \frac{\ln(250/250) + (0.10 - 0.03 + 0.18^2/2)0.25}{0.18\sqrt{0.25}} = 0.2394$$

$$d_2 = d_1 - 0.18\sqrt{0.25} = 0.1494$$

and the call price is

$$250N(0.2394)e^{-0.03\times 0.25} - 250N(0.1494)e^{-0.10\times 0.25}$$

$$= 250 \times 0.5946e^{-0.03\times 0.25} - 250 \times 0.5594e^{-0.10\times 0.25}$$

or 11.15.

Problem 15.7.

Calculate the value of an eight-month European put option on a currency with a strike price of 0.50. The current exchange rate is 0.52, the volatility of the exchange rate is 12%, the domestic risk-free interest rate is 4% per annum, and the foreign risk-free interest rate is 8% per annum.

In this case $S_0 = 0.52$, $K = 0.50$, $r = 0.04$, $r_f = 0.08$, $\sigma = 0.12$, $T = 0.6667$, and

$$d_1 = \frac{\ln(0.52/0.50) + (0.04 - 0.08 + 0.12^2/2)0.6667}{0.12\sqrt{0.6667}} = 0.1771$$

$$d_2 = d_1 - 0.12\sqrt{0.6667} = 0.0791$$

and the put price is

$$0.50N(-0.0791)e^{-0.04\times 0.6667} - 0.52N(-0.1771)e^{-0.08\times 0.6667}$$

$$= 0.50 \times 0.4685e^{-0.04\times 0.6667} - 0.52 \times 0.4297e^{-0.08\times 0.6667}$$

$$= 0.0162$$

Problem 15.8.

Show that the formula in equation (15.12) for a put option to sell one unit of currency A for currency B at strike price K gives the same value as equation (15.11) for a call option to buy K units of currency B for currency A at a strike price of 1/K.

A put option to sell one unit of currency A for K units of currency B is worth

$$Ke^{-r_B T}N(-d_2) - S_0 e^{-r_A T}N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r_A - r_B + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r_A - r_B - \sigma^2/2)T}{\sigma\sqrt{T}}$$

and r_A and r_B are the risk-free rates in currencies A and B, respectively. The value of the option is measured in units of currency B. Defining $S_0^* = 1/S_0$ and $K^* = 1/K$

$$d_1 = \frac{-\ln(S_0^*/K^*) - (r_A - r_B - \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{-\ln(S_0^*/K^*) - (r_A - r_B + \sigma^2/2)T}{\sigma\sqrt{T}}$$

The put price is therefore

$$S_0 K[S_0^* e^{-r_B T}N(d_1^*) - K^* e^{-r_A T}N(d_1^*)$$

where

$$d_1^* = \frac{\ln(S_0^*/K^*) + (r_B - r_A - \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2^* = \frac{\ln(S_0^*/K^*) + (r_B - r_A + \sigma^2/2)T}{\sigma\sqrt{T}}$$

This shows that put option is equivalent to KS_0 call options to buy 1 unit of currency A for $1/K$ units of currency B. In this case the value of the option is measured in units of currency A. To obtain the call option value in units of currency B (the same units as the value of the put option was measured in) we must divide by S_0. This proves the result.

Problem 15.9.

A foreign currency is currently worth \$1.50. The domestic and foreign risk-free interest rates are 5% and 9%, respectively. Calculate a lower bound for the value of a six-month call option on the currency with a strike price of \$1.40 if it is (a) European and (b) American.

Lower bound for European option is

$$S_0 e^{-r_f T} - Ke^{-rT} = 1.5e^{-0.09\times 0.5} - 1.4e^{-0.05\times 0.5} = 0.069$$

Lower bound for American option is

$$S_0 - K = 0.10$$

Problem 15.10.

Consider a stock index currently standing at 250. The dividend yield on the index is 4% per annum, and the risk-free rate is 6% per annum. A three-month European call option on the index with a strike price of 245 is currently worth \$10. What is the value of a three-month put option on the index with a strike price of 245?

In this case $S_0 = 250$, $q = 0.04$, $r = 0.06$, $T = 0.25$, $K = 245$, and $c = 10$. Using put–call parity

$$c + Ke^{-rT} = p + S_0e^{-qT}$$

or

$$p = c + Ke^{-rT} - S_0e^{-qT}$$

Substituting:

$$p = 10 + 245e^{-0.25\times0.06} - 250e^{-0.25\times0.04} = 3.84$$

The put price is 3.84.

Problem 15.11.

An index currently stands at 696 and has a volatility of 30% per annum. The risk-free rate of interest is 7% per annum and the index provides a dividend yield of 4% per annum. Calculate the value of a three-month European put with an exercise price of 700.

In this case $S_0 = 696$, $K = 700$, $r = 0.07$, $\sigma = 0.3$, $T = 0.25$ and $q = 0.04$. The option can be valued using equation (15.5).

$$d_1 = \frac{\ln(696/700) + (0.07 - 0.04 + 0.09/2) \times 0.25}{0.3\sqrt{0.25}} = 0.0868$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = -0.0632$$

and

$$N(-d_1) = 0.4654, \quad N(-d_2) = 0.5252$$

The value of the put, p, is given by:

$$p = 700e^{-0.07\times0.25} \times 0.5252 - 696e^{-0.04\times0.25} \times 0.4654 = 40.6$$

i.e., it is \$40.6.

Problem 15.12.

Show that if C is the price of an American call with exercise price K and maturity T on a stock paying a dividend yield of q, and P is the price of an American put on the same stock with the same strike price and exercise date,

$$S_0e^{-qT} - K < C - P < S_0 - Ke^{-rT}$$

where S_0 is the stock price, r is the risk-free rate, and $r > 0$. (Hint: To obtain the first half of the inequality, consider possible values of:

Portfolio A: a European call option plus an amount K invested at the risk-free rate
Portfolio B: an American put option plus e^{-qT} of stock with dividends being reinvested in the stock
To obtain the second half of the inequality, consider possible values of:
Portfolio C: an American call option plus an amount Ke^{-rT} invested at the risk-free rate
Portfolio D: a European put option plus one stock with dividends being reinvested in the stock)

Following the hint, we first consider

Portfolio A: A European call option plus an amount K invested at the risk-free rate

Portfolio B: An American put option plus e^{-qT} of stock with dividends being reinvested in the stock.

Portfolio A is worth $c + K$ while portfolio B is worth $P + S_0e^{-qT}$. If the put option is exercised at time τ $(0 \leq \tau < T)$, portfolio B becomes:

$$K - S_\tau + S_\tau e^{-q(T-\tau)} \leq K$$

where S_τ is the stock price at time τ. Portfolio A is worth

$$c + Ke^{r\tau} \geq K$$

Hence portfolio A is worth at least as much as portfolio B. If both portfolios are held to maturity (time T), portfolio A is worth

$$\begin{aligned} &\max(S_T - K, 0) + Ke^{rT} \\ =&\max(S_T, K) + K(e^{rT} - 1) \end{aligned}$$

Portfolio B is worth $\max(S_T, K)$. Hence portfolio A is worth more than portfolio B.

Because portfolio A is worth at least as much as portfolio B in all circumstances

$$P + S_0e^{-qT} \leq c + K$$

Because $c \leq C$:

$$P + S_0e^{-qT} \leq C + K$$

or

$$S_0e^{-qT} - K \leq C - P$$

This proves the first part of the inequality.

For the second part consider:

Portfolio C: An American call option plus an amount Ke^{-rT} invested at the risk-free rate

Portfolio D: A European put option plus one stock with dividends being reinvested in the stock.

Portfolio C is worth $C + Ke^{-rT}$ while portfolio D is worth $p + S_0$. If the call option is exercised at time τ $(0 \le \tau < T)$ portfolio C becomes:

$$S_\tau - K + Ke^{-r(T-\tau)} < S_\tau$$

while portfolio D is worth

$$p + S_\tau e^{q(\tau-t)} \ge S_\tau$$

Hence portfolio D is worth more than portfolio C. If both portfolios are held to maturity (time T), portfolio C is worth $\max(S_T, K)$ while portfolio D is worth

$$\begin{aligned} &\max(K - S_T, 0) + S_T e^{qT} \\ &= \max(S_T, K) + S_T(e^{qT} - 1) \end{aligned}$$

Hence portfolio D is worth at least as much as portfolio C.

Since portfolio D is worth at least as much as portfolio C in all circumstances:

$$C + Ke^{-rT} \le p + S_0$$

Since $p \le P$:

$$C + Ke^{-rT} \le P + S_0$$

or

$$C - P \le S_0 - Ke^{-rT}$$

This proves the second part of the inequality. Hence:

$$S_0 e^{-qT} - K \le C - P \le S_0 - Ke^{-rT}$$

Problem 15.13.

Show that a European call option on a currency has the same price as the corresponding European put option on the currency when the forward price equals the strike price.

This follows from put–call parity and the relationship between the forward price, F_0, and the spot price, S_0.

$$c + Ke^{-rT} = p + S_0 e^{-r_f T}$$

and

$$F_0 = S_0 e^{(r-r_f)T}$$

so that

$$c + Ke^{-rT} = p + F_0 e^{-rT}$$

If $K = F_0$ this reduces to $c = p$. The result that $c = p$ when $K = F_0$ is true for options on all underlying assets, not just options on currencies. An at-the-money option is frequently defined as one where $K = F_0$ (or $c = p$) rather than one where $K = S_0$.

Problem 15.14.

Would you expect the volatility of a stock index to be greater or less than the volatility of a typical stock? Explain your answer.

The volatility of a stock index can be expected to be less than the volatility of a typical stock. This is because some risk (i.e., return uncertainty) is diversified away when a portfolio of stocks is created. In capital asset pricing model terminology, there exists systematic and unsystematic risk in the returns from an individual stock. However, in a stock index, unsystematic risk has been diversified away and only the systematic risk contributes to volatility.

Problem 15.15.

Does the cost of portfolio insurance increase or decrease as the beta of a portfolio increases? Explain your answer.

The cost of portfolio insurance increases as the beta of the portfolio increases. This is because portfolio insurance involves the purchase of a put option on the portfolio. As beta increases, the volatility of the portfolio increases causing the cost of the put option to increase. When index options are used to provide portfolio insurance, both the number of options required and the strike price increase as beta increases.

Problem 15.16.

Suppose that a portfolio is worth \$60 million and the S&P 500 is at 1200. If the value of the portfolio mirrors the value of the index, what options should be purchased to provide protection against the value of the portfolio falling below \$54 million in one year's time?

If the value of the portfolio mirrors the value of the index, the index can be expected to have dropped by 10% when the value of the portfolio drops by 10%. Hence when the value of the portfolio drops to \$54 million the value of the index can be expected to be 1080. This indicates that put options with an exercise price of 1080 should be purchased. The options should be on:

$$\frac{60,000,000}{1200} = \$50,000$$

times the index. Each option contract is for \$100 times the index. Hence 500 contracts should be purchased.

Problem 15.17.

Consider again the situation in Problem 15.16. Suppose that the portfolio has a beta of 2.0, the risk-free interest rate is 5% per annum, and the dividend yield on both the portfolio and the index is 3% per annum. What options should be purchased to provide protection against the value of the portfolio falling below \$54 million in one year's time?

When the value of the portfolio falls to \$54 million the holder of the portfolio makes a capital loss of 10%. After dividends are taken into account the loss is 7% during the year. This is 12% below the risk-free interest rate. According to the capital asset pricing model:

$$\begin{array}{c}\text{Excess expected return of portfolio}\\ \text{above riskless interest rate}\end{array} = \beta \times \begin{array}{c}\text{Excess expected return of market}\\ \text{above riskless interest rate}\end{array}$$

Therefore, when the portfolio provides a return 12% below the risk-free interest rate, the market's expected return is 6% below the risk-free interest rate. As the index can be assumed to have a beta of 1.0, this is also the excess expected return (including dividends) from the index. The expected return from the index is therefore -1% per annum. Since the index provides a 3% per annum dividend yield, the expected movement in the index is -4%. Thus when the portfolio's value is \$54 million the expected value of the index $0.96 \times 1200 = 1152$. Hence European put options should be purchased with an exercise price of 1152. Their maturity date should be in one year.

The number of options required is twice the number required in Problem 15.16. This is because we wish to protect a portfolio which is twice as sensitive to changes in market conditions as the portfolio in Problem 15.16. Hence options on \$100,000 (or 1,000 contracts) should be purchased. To check that the answer is correct consider what happens when the value of the portfolio declines by 20% to \$48 million. The return including dividends is -17%. This is 22% less than the risk-free interest rate. The index can be expected to provide a return (including dividends) which is 11% less than the risk-free interest rate, i.e. a return of -6%. The index can therefore be expected to drop by 9% to 1092. The payoff from the put options is $(1152 - 1092) \times 100,000 =$ \$6 million. This is exactly what is required to restore the value of the portfolio to \$54 million.

Problem 15.18.

An index currently stands at 1,500. European call and put options with a strike price of 1,400 and time to maturity of six months have market prices of 154.00 and 34.25, respectively. The six-month risk-free rate is 5%. What is the implied dividend yield?

The implied dividend yield is the value of q that satisfies the put–call parity equation. It is the value of q that solves

$$154 + 1400e^{-0.05 \times 0.5} = 34.25 + 1500e^{-0.5q}$$

This is 1.99%.

Problem 15.19.

A total return index tracks the return, including dividends, on a certain portfolio. Explain how you would value (a) forward contracts and (b) European options on the index.

A total return index behaves like a stock paying no dividends. In a risk-neutral world it can be expected to grow on average at the risk-free rate. Forward contracts and options on total return indices should be valued in the same way as forward contracts and options on non-dividend-paying stocks.

Problem 15.20.

What is the put–call parity relationship for European currency options

The put–call parity relationship for European currency options is

$$c + Ke^{-rT} = p + Se^{-r_f T}$$

To prove this result, the two portfolios to consider are:

Portfolio A: one call option plus one discount bond which will be worth K at time T

Portfolio B: one put option plus $e^{-r_f T}$ of foreign currency invested at the foreign risk-free interest rate.

Both portfolios are worth $\max(S_T, K)$ at time T. They must therefore be worth the same today. The result follows.

Problem 15.21.

Can an option on the yen-euro exchange rate be created from two options, one on the dollar-euro exchange rate, and the other on the dollar-yen exchange rate? Explain your answer.

There is no way of doing this. A natural idea is to create an option to exchange K euros for one yen from an option to exchange Y dollars for 1 yen and an option to exchange K euros for Y dollars. The problem with this is that it assumes that either both options are exercised or that neither option is exercised. There are always some circumstances where the first option is in-the-money at expiration while the second is not and vice versa.

Problem 15.22.

Prove the results in equation (15.1), (15.2), and (15.3) using the portfolios indicated.

In portfolio A, the cash, if it is invested at the risk-free interest rate, will grow to K at time T. If $S_T > K$, the call option is exercised at time T and portfolio A is worth S_T. If $S_T < K$, the call option expires worthless and the portfolio is worth K. Hence, at time T, portfolio A is worth

$$\max(S_T, K)$$

Because of the reinvestment of dividends, portfolio B becomes one share at time T. It is, therefore, worth S_T at this time. It follows that portfolio A is always worth as much as, and is sometimes worth more than, portfolio B at time T. In the absence of arbitrage opportunities, this must also be true today. Hence,

$$c + Ke^{-rT} \geq S_0 e^{-qT}$$

or

$$c \geq S_0 e^{-qT} - Ke^{-rT}$$

This proves equation (15.1)

In portfolio C, the reinvestment of dividends means that the portfolio is one put option plus one share at time T. If $S_T < K$, the put option is exercised at time T and portfolio C is worth K. If $S_T > K$, the put option expires worthless and the portfolio is worth S_T. Hence, at time T, portfolio C is worth

$$\max(S_T, K)$$

Portfolio D is worth K at time T. It follows that portfolio C is always worth as much as, and is sometimes worth more than, portfolio D at time T. In the absence of arbitrage opportunities, this must also be true today. Hence,

$$p + S_0 e^{-qT} \geq K e^{-rT}$$

or

$$p \geq K e^{-rT} - S_0 e^{-qT}$$

This proves equation (15.2)

Portfolios A and C are both worth $\max(S_T, K)$ at time T. They must, therefore, be worth the same today, and the put–call parity result in equation (15.3) follows.

ASSIGNMENT QUESTIONS

Problem 15.23.

The Dow Jones Industrial Average on January 12, 2007 was 12,556 and the price of the March 126 call was \$2.25. Use the DerivaGem software to calculate the implied volatility of this option. Assume that the risk-free rate was 5.3% and the dividend yield was 3%. The option expires on March 20, 2007. Estimate the price of a March 126 put. What is the volatility implied by the price you estimate for this option? (Note that options are on the Dow Jones index divided by 100.

Options on the DJIA are European. There are 47 trading days between January 12, 2007 and March 20, 2007. Setting the time to maturity equal to 47/252 = 0.1865, DerivaGem gives the implied volatility as 10.23%. (If instead we use calendar days the time to maturity is 67/365=0.1836 and the implied volatility is 10.33%.)

From put call parity (equation 13.3) the price of the put, p, (using trading time) is given by

$$2.25 + 126e^{-0.053\times 0.1865} = p + 125.56e^{-0.03\times 0.1865}$$

so that $p = 2.1512$. DerivaGem shows that the implied volatility is 10.23% (as for the call). (If calendar time is used the price of the put is 2.1597 and the implied volatility is 10.33% as for the call.)

A European call has the same implied volatility as a European put when both have the same strike price and time to maturity. This is formally proved in the appendix to Chapter 17.

Problem 15.24.

A stock index currently stands at 300 and has a volatility of 20%. The risk-free interest rate is 8% and the dividend yield on the index is 3%. Use a three-step binomial tree to value a six-month put option on the index with a strike price of 300 if it is (a) European and (b) American?

A shown by DerivaGem the value of the European option is 14.39 and the value of the American option is 14.97.

Problem 15.25.

Suppose that the spot price of the Canadian dollar is U.S. $0.85 and that the Canadian dollar/U.S. dollar exchange rate has a volatility of 4% per annum. The risk-free rates of interest in Canada and the United States are 4% and 5% per annum, respectively. Calculate the value of a European call option to buy one Canadian dollar for U.S. $0.85 in nine months. Use put-call parity to calculate the price of a European put option to sell one Canadian dollar for U.S. $0.85 in nine months. What is the price of a call option to buy U.S. $0.85 with one Canadian dollar in nine months?

In this case $S_0 = 0.85$, $K = 0.85$, $r = 0.05$, $r_f = 0.04$, $\sigma = 0.04$ and $T = 0.75$. The option can be valued using equation (15.11)

$$d_1 = \frac{\ln(0.85/0.85) + (0.05 - 0.04 + 0.0016/2) \times 0.75}{0.04\sqrt{0.75}} = 0.2338$$

$$d_2 = d_1 - 0.04\sqrt{0.75} = 0.1992$$

and

$$N(d_1) = 0.5924, \quad N(d_2) = 0.5789$$

The value of the call, c, is given by

$$c = 0.85e^{-0.04\times0.75} \times 0.0.5924 - 0.85e^{-0.05\times0.75} \times 0.5789 = 0.0147$$

i.e., it is 1.47 cents. From put–call parity

$$p + S_0e^{-r_fT} = c + Ke^{-rT}$$

so that

$$p = 0.0147 + 0.85e^{-0.05\times9/12} - 0.85e^{-0.04\times9/12} = 0.00854$$

The option to buy US$0.85 with C$1.00 is the same as the same as an option to sell one Canadian dollar for US$0.85. This means that it is a put option on the Canadian dollar and its price is US$0.00854.

Problem 15.26.

A mutual fund announces that the salaries of its fund managers will depend on the performance of the fund. If the fund loses money, the salaries will be zero. If the fund makes a profit, the salaries will be proportional to the profit. Describe the salary of a fund manager as an option. How is a fund manager motivated to behave with this type of remuneration package?

Suppose that K is the value of the fund at the beginning of the year and S_T is the value of the fund at the end of the year.

The salary of a fund manager is

$$\alpha \max(S_T - K, 0)$$

where α is a constant.

This shows that a fund manager has a call option on the value of the fund at the end of the year. All of the parameters determining the value of this call option are outside the control of the fund manager except the volatility of the fund. The fund manager has an incentive to make the fund as volatile as possible! This may not correspond with the desires of the investors. One way of making the fund highly volatile would be by investing only in high-beta stocks. Another would be by using the whole fund to buy call options on a market index.

It might be argued that a fund manager would not do this because of the risk which the manager faces. If the fund earns a negative return the manager's salary is zero. However, a fund manager could hedge the risk of a negative return by, on his or her own account, taking a short position in call options on a stock market index.

The position could be chosen so that if the market goes up, the gain on salary more than offsets the losses on the call options.

If the market goes down the fund manager ends up with the price received for the call options. It is easy to see that the strategy becomes more attractive as the riskiness of the fund's portfolio increases.

To summarize, the (superficially attractive) remuneration package is open to abuse and does not necessarily motivate the fund managers to act in the best interests of the fund's investors.

Problem 15.27.

Assume that the price of currency A expressed in terms of the price of currency B follows the process

$$dS = (r_{\mathrm{B}} - r_{\mathrm{A}})S\,dt + \sigma S\,dz$$

where r_{A} is the risk-free interest rate in currency A and r_{B} is the risk-free interest rate in currency B. What is the process followed by the price of currency B expressed in terms of currency A?

The price of currency B expressed in terms of currency A is $1/S$. From Ito's lemma the process followed by $X = 1/S$ is

$$dX = [(r_{\mathrm{B}} - r_{\mathrm{A}})S \times (-1/S^2) + 0.5\sigma^2 S^2 \times (2/S^3)]dt + \sigma S \times (-1/S^2)dz$$

or

$$dX = [r_{\mathrm{A}} - r_{\mathrm{B}} + \sigma^2]X dt - \sigma X dz$$

This is Siegel's paradox and is discussed further in Business Snapshot 29.1.

Problem 15.28.

The three-month forward USD/euro exchange rate is 1.3000. The exchange rate volatility is 15%. A US company will have to pay 1 million euros in three months. The *euro and USD risk-free rates are 5% and 4%, respectively. The company decides to use a range forward contract with the lower strike price equal to 1.2500.*

(a) What should the higher strike price be to create a zero-cost contract.

(b) What position in calls and puts should the company take.

(c) Does your answer depend on the euro risk-free rate? Explain.

(d) Does your answer depend on the USD risk-free rate? Explain.

(a) A put with a strike price of 1.25 is worth \$0.019. By trial and error DerivaGem can be used to show that the strike price of a call that leads to a call having a price of \$0.019 is 1.3477. This is the higher strike price to create a zero cost contract.

(b) The company should sell a put with strike price 1.25 and buy a call with strike price 1.3477. This ensures that the exchange rate it pays for the euros is between 1.2500 and 1.3477.

(c) The answer does depend on the euro risk-free rate because the forward exchange rate depends on this rate

(d) The answer does depend on the dollar risk-free rate because the forward exchange rate depends on this rate. However, if the interest rates change so that the spread between the dollar and euro interest rates remains the same, the upper strike price is unchanged at 1.3477. This can be seen from equations (15.13) and (15.14). The forward exchange rate, F_0 is unchanged and changing r has the same percentage effect on both the call and the put.

CHAPTER 16
Futures Options

Notes for the Instructor

The chapter concerned with options on stock indices, currencies, and futures in the sixth edition has been split into two chapters (15 and 16) in the seventh edition. Chapter 15 is concerned with options on stock indices and currencies; Chapter 16 is concerned with options on futures.

The material on futures options has been restructured for the seventh edition. The chapter now spends more time discussing how Black's model can be used to price European options in terms of forward or futures prices. This is important material because in practice it is usually the case that practitioners use Black's model rather than Black-Scholes model for European options. By doing this they avoid the need to estimate the income on the underlying asset explicitly. (The Black's model material in this chapter is extended to the stochastic interest rate case in Section 27.6.) This chapter also discusses futures style options which are becoming popular at some exchanges. A futures style option is a futures contract on the payoff from an option.

The way I approach the material in the chapter is indicated by the slides. I like to use Problem 16.23 as a hand-in assignment because it provides practice using DerivaGem and links in with the material on volatility smiles in Chapter 18 and the material on American options in Chapter 19.

QUESTIONS AND PROBLEMS

Problem 16.1

Explain the difference between a call option on yen and a call option on yen futures.

A call option on yen gives the holder the right to buy yen in the spot market at an exchange rate equal to the strike price. A call option on yen futures gives the holder the right to receive the amount by which the futures price exceeds the strike price. If the yen futures option is exercised, the holder also obtains a long position in the yen futures contract.

Problem 16.2.

Why are options on bond futures more actively traded than options on bonds?

The main reason is that a bond futures contract is a more liquid instrument than a bond. The price of a Treasury bond futures contract is known immediately from trading on CBOT. The price of a bond can be obtained only by contacting dealers.

Problem 16.3.

"A futures price is like a stock paying a dividend yield." What is the dividend yield?

A futures price behaves like a stock paying a dividend yield at the risk-free interest rate.

Problem 16.4.

A futures price is currently 50. At the end of six months it will be either 56 or 46. The risk-free interest rate is 6% per annum. What is the value of a six-month European call option with a strike price of 50?

In this case $u = 1.12$ and $d = 0.92$. The probability of an up movement in a risk-neutral world is

$$\frac{1 - 0.92}{1.12 - 0.92} = 0.4$$

From risk-neutral valuation, the value of the call is

$$e^{-0.06\times0.5}(0.4 \times 6 + 0.6 \times 0) = 2.33$$

Problem 16.5.

How does the put–call parity formula for a futures option differ from put–call parity for an option on a non-dividend-paying stock?

The put–call parity formula for futures options is the same as the put–call parity formula for stock options except that the stock price is replaced by F_0e^{-rT}, where F_0 is the current futures price, r is the risk-free interest rate, and T is the life of the option.

Problem 16.6.

Consider an American futures call option where the futures contract and the option contract expire at the same time. Under what circumstances is the futures option worth more than the corresponding American option on the underlying asset?

The American futures call option is worth more than the corresponding American option on the underlying asset when the futures price is greater than the spot price prior to the maturity of the futures contract. This is the case when the risk-free rate is greater than the income on the asset plus the convenience yield.

Problem 16.7.

Calculate the value of a five-month European put futures option when the futures price is \$19, the strike price is \$20, the risk-free interest rate is 12% per annum, and the volatility of the futures price is 20% per annum.

In this case $F_0 = 19$, $K = 20$, $r = 0.12$, $\sigma = 0.20$, and $T = 0.4167$. The value of the European put futures option is

$$20N(-d_2)e^{-0.12\times0.4167} - 19N(-d_1)e^{-0.12\times0.4167}$$

where

$$d_1 = \frac{\ln(19/20) + (0.04/2)0.4167}{0.2\sqrt{0.4167}} = -0.3327$$

$$d_2 = d_1 - 0.2\sqrt{0.4167} = -0.4618$$

This is

$$e^{-0.12\times0.4167}[20N(0.4618) - 19N(0.3327)]$$

$$= e^{-0.12\times0.4167}(20 \times 0.6778 - 19 \times 0.6303)$$

$$= 1.50$$

or \$1.50.

Problem 16.8.

Suppose you buy a put option contract on October gold futures with a strike price of \$700 per ounce. Each contract is for the delivery of 100 ounces. What happens if you exercise when the October futures price is \$680?

An amount $(700 - 680) \times 100 = \$2,000$ is added to your margin account and you acquire a short futures position giving you the right to sell 100 ounces of gold in October. This position is marked to market in the usual way until you choose to close it out.

Problem 16.9.

Suppose you sell a call option contract on April live cattle futures with a strike price of 90 cents per pound. Each contract is for the delivery of 40,000 pounds. What happens if the contract is exercised when the futures price is 95 cents?

In this case an amount $(0.95 - 0.90) \times 40,000 = \$2,000$ is subtracted from your margin account and you acquire a short position in a live cattle futures contract to sell 40,000 pounds of cattle in April. This position is marked to marked in the usual way until you choose to close it out.

Problem 16.10.

Consider a two-month call futures option with a strike price of 40 when the risk-free interest rate is 10% per annum. The current futures price is 47. What is a lower bound for the value of the futures option if it is (a) European and (b) American?

Lower bound if option is European is

$$(F_0 - K)e^{-rT} = (47 - 40)e^{-0.1\times2/12} = 6.88$$

Lower bound if option is American is

$$F_0 - K = 7$$

Problem 16.11.

Consider a four-month put futures option with a strike price of 50 when the risk-free interest rate is 10% per annum. The current futures price is 47. What is a lower bound for the value of the futures option if it is (a) European and (b) American?

Lower bound if option is European is

$$(K - F_0)e^{-rT} = (50 - 47)e^{-0.1\times4/12} = 2.90$$

Lower bound if option is American is

$$K - F_0 = 3$$

Problem 16.12.

A futures price is currently 60 and its volatility is 30%. The risk-free interest rate is 8% per annum. Use a two-step binomial tree to calculate the value of a six-month call option on the futures with a strike price of 60 If the call were American, would it ever be worth exercising it early?

In this case $u = e^{0.3\times\sqrt{0.25}} = 1.1618$ and $d = 1/u = 0.8607$ the risk-neutral probability of an up move is

$$p = \frac{1 - 0.8607}{1.1618 - 0.8607} = 0.4626$$

In the tree shown in Figure S16.1 the middle number at each node is the price of the European option and the lower number is the price of the American option. The tree shows that the price of the European option is 4.3155 and the price of the American option is 4.4026. The American option should sometimes be exercised early.

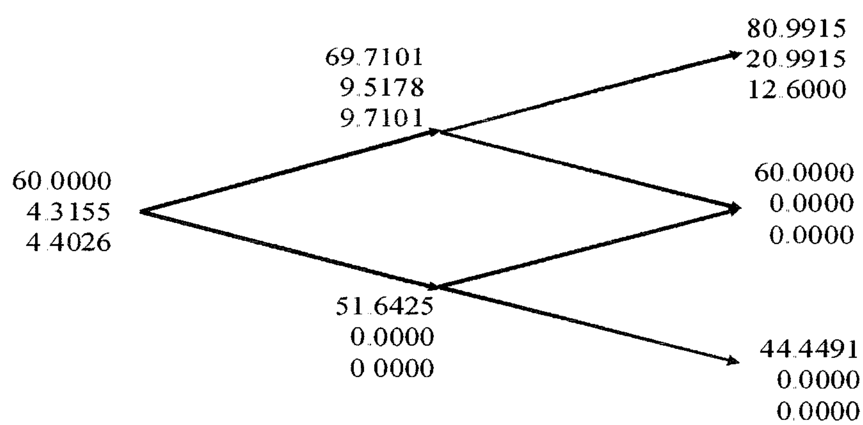


Figure S16.1 Tree to evaluate European and American call options in Problem 16.12.

Problem 16.13.

In Problem 16.12 what is the value of a six-month European put option on futures with a strike price of 60? If the put were American, would it ever be worth exercising it early? Verify that the call prices calculated in Problem 16.12 and the put prices calculated here satisfy put–call parity relationships.

The parameters u, d, and p are the same as in Problem 16.12. The tree in Figure S16.2 shows that the price of the European option is 3.0265 while the price of the American option is 3.0847.

Because $c = p$ and $F_0 = K$ the put–call parity realtionship in equation (16.1) clearly holds. For the American option prices we have:

$$C - P = 0; \qquad F_0e^{-rT} - K = -2.353; \qquad F_0 - Ke^{-rT} = 2.353$$

The put–call inequalities for American options in equation (16.2) are therefore satisfied

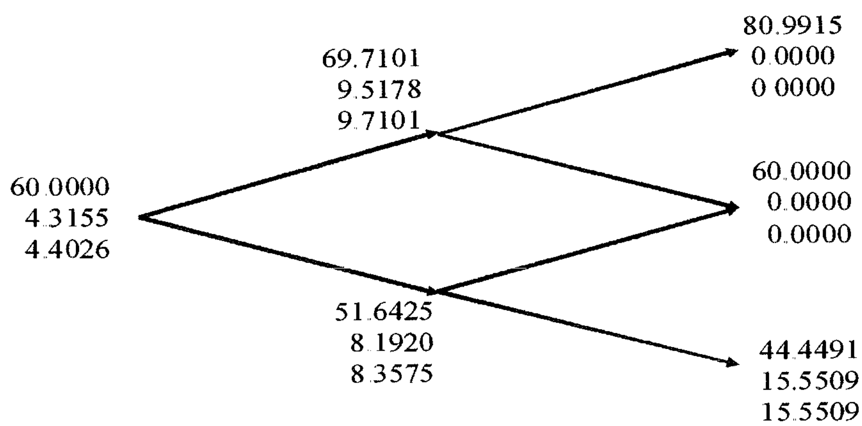


Figure S16.2 Tree to evaluate European and American put options in Problem 16.13.

Problem 16.14.

A futures price is currently 25, its volatility is 30% per annum, and the risk-free interest rate is 10% per annum. What is the value of a nine-month European call on the futures with a strike price of 26?

In this case $F_0 = 25$, $K = 26$, $\sigma = 0.3$, $r = 0.1$, $T = 0.75$

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}} = -0.0211$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = -0.2809$$

$$c = e^{-0.075}[25N(-0.0211) - 26N(-0.2809)]$$

$$= e^{-0.075}[25 \times 0.4916 - 26 \times 0.3894] = 2.01$$

Problem 16.15.

A futures price is currently 70, its volatility is 20% per annum, and the risk-free interest rate is 6% per annum. What is the value of a five-month European put on the futures with a strike price of 65?

In this case $F_0 = 70$, $K = 65$, $\sigma = 0.2$, $r = 0.06$, $T = 0.4167$

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}} = 0.6386$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = 0.5095$$

$$p = e^{-0.025}[65N(-0.5095) - 70N(-0.6386)]$$
$$= e^{-0.025}[65 \times 0.3052 - 70 \times 0.2615] = 1.495$$

Problem 16.16.

Suppose that a one-year futures price is currently 35. A one-year European call option and a one-year European put option on the futures with a strike price of 34 are both priced at 2 in the market. The risk-free interest rate is 10% per annum. Identify an arbitrage opportunity.

In this case

$$c + Ke^{-rT} = 2 + 34e^{-0.1\times 1} = 32.76$$
$$p + F_0 e^{-rT} = 2 + 35e^{-0.1\times 1} = 33.67$$

Put-call parity shows that we should buy one call, short one put and short a futures contract. This costs nothing up front. In one year, either we exercise the call or the put is exercised against us. In either case, we buy the asset for 34 and close out the futures position. The gain on the short futures position is $35 - 34 = 1$.

Problem 16.17.

"The price of an at-the-money European call futures option always equals the price of a similar at-the-money European put futures option." Explain why this statement is true.

The put price is

$$e^{-rT}[KN(-d_2) - F_0 N(-d_1)]$$

Because $N(-x) = 1 - N(x)$ for all x the put price can also be written

$$e^{-rT}[K - KN(d_2) - F_0 + F_0 N(d_1)]$$

Because $F_0 = K$ this is the same as the call price:

$$e^{-rT}[F_0 N(d_1) - KN(d_2)]$$

This result can also be proved from put–call parity showing that it is not model dependent.

Problem 16.18.

Suppose that a futures price is currently 30. The risk-free interest rate is 5% per annum. A three-month American call futures option with a strike price of 28 is worth 4. Calculate bounds for the price of a three-month American put futures option with a strike price of 28.

From equation (16.2), $C - P$ must lie between

$$30e^{-0.05\times 3/12} - 28 = 1.63$$

and

$$30 - 28e^{-0.05\times 3/12} = 2.35$$

Because $C = 4$ we must have $1.63 < 4 - P < 2.35$ or

$$1.65 < P < 2.37$$

Problem 16.19.

Show that if C is the price of an American call option on a futures contract when the strike price is K and the maturity is T, and P is the price of an American put on the same futures contract with the same strike price and exercise date,

$$F_0e^{-rT} - K < C - P < F_0 - Ke^{-rT}$$

where F_0 is the futures price and r is the risk-free rate. Assume that $r > 0$ and that there is no difference between forward and futures contracts. (Hint: Use an analogous approach to that indicated for Problem 15.12.)

In this case we consider

Portfolio A: A European call option on futures plus an amount K invested at the risk-free interest rate

Portfolio B: An American put option on futures plus an amount F_0e^{-rT} invested at the risk-free interest rate plus a long futures contract maturing at time T.

Following the arguments in Chapter 5 we will treat all futures contracts as forward contracts. Portfolio A is worth $c + K$ while portfolio B is worth $P + F_0e^{-rT}$. If the put option is exercised at time τ $(0 \le \tau < T)$, portfolio B is worth

$$K - F_\tau + F_0e^{-r(T-\tau)} + F_\tau - F_0$$
$$= K + F_0e^{-r(T-\tau)} - F_0 < K$$

at time τ where F_τ is the futures price at time τ. Portfolio A is worth

$$c + Ke^{r\tau} \ge K$$

Hence Portfolio A more than Portfolio B. If both portfolios are held to maturity (time T), Portfolio A is worth

$$\begin{aligned}&\max(F_T - K, 0) + Ke^{rT}\\ &= \max(F_T, K) + K(e^{rT} - 1)\end{aligned}$$

Portfolio B is worth

$$\max(K - F_T, 0) + F_0 + F_T - F_0 = \max(F_T, K)$$

Hence portfolio A is worth more than portfolio B.

Because portfolio A is worth more than portfolio B in all circumstances:

$$P + F_0 e^{-r(T-t)} < c + K$$

Because $c \leq C$ it follows that

$$P + F_0 e^{-rT} < C + K$$

or

$$F_0 e^{-rT} - K < C - P$$

This proves the first part of the inequality.

For the second part of the inequality consider:

Portfolio C: An American call futures option plus an amount Ke^{-rT} invested at the risk-free interest rate

Portfolio D: A European put futures option plus an amount F_0 invested at the risk-free interest rate plus a long futures contract.

Portfolio C is worth $C + Ke^{-rT}$ while portfolio D is worth $p + F_0$. If the call option is exercised at time τ $(0 \leq \tau < T)$ portfolio C becomes:

$$F_\tau - K + Ke^{-r(T-\tau)} < F_\tau$$

while portfolio D is worth

$$\begin{aligned}&p + F_0 e^{r\tau} + F_\tau - F_0\\ &= p + F_0(e^{r\tau} - 1) + F_\tau \geq F_\tau\end{aligned}$$

Hence portfolio D is worth more than portfolio C. If both portfolios are held to maturity (time T), portfolio C is worth $\max(F_T, K)$ while portfolio D is worth

$$\begin{aligned}&\max(K - F_T, 0) + F_0 e^{rT} + F_T - F_0\\ &= \max(K, F_T) + F_0(e^{rT} - 1)\\ &> \max(K, F_T)\end{aligned}$$

Hence portfolio D is worth more than portfolio C.

Because portfolio D is worth more than portfolio C in all circumstances

$$C + Ke^{-rT} < p + F_0$$

Because $p \leq P$ it follows that

$$C + Ke^{-rT} < P + F_0$$

or

$$C - P < F_0 - Ke^{-rT}$$

This proves the second part of the inequality. The result:

$$F_0 e^{-rT} - K < C - P < F_0 - Ke^{-rT}$$

has therefore been proved.

Problem 16.20.

Calculate the price of a three-month European call option on the spot price of silver. The three-month futures price is \$12, the strike price is \$13, the risk-free rate is 4%, and the volatility of the price of silver is 25%.

This has the same value as a three-month call option on silver futures where the futures contract expires in three months. It can therefore be valued using equation (16.9) with $F_0 = 12$, $K = 13$, $r = 0.04$, $\sigma = 0.25$ and $T = 0.25$. The value is 0.244.

Problem 16.21.

A corporation knows that in three months it will have \$5 million to invest for 90 days at LIBOR minus 50 basis points and wishes to ensure that the rate obtained will be at least 6.5%. What position in exchange-traded interest-rate options should the corporation take?

The rate received will be less than 6.5% when LIBOR is less than 7%. The corporation requires a three-month call option on a Eurodollar futures option with a strike price of 93. If three-month LIBOR is greater than 7% at the option maturity, the Eurodollar futures quote at option maturity will be less than 93 and there will be no payoff from the option. If the three-month LIBOR is less than 7%, one Eurodollar futures options provide a payoff of \$25 per 0.01%. Each 0.01% of interest costs the corporation \$500 ($= 5,000,000 \times 0.0001$). A total of $500/25 = 20$ contracts are therefore required.

ASSIGNMENT QUESTIONS

Problem 16.22.

A futures price is currently 40. It is known that at the end of three months the price will be either 35 or 45. What is the value of a three-month European call option on the futures with a strike price of 42 if the risk-free interest rate is 7% per annum?

In this case $u = 1.125$ and $d = 0.875$. The risk-neutral probability of an up move is

$$(1 - .875)/(1.125 - 0.875) = 0.5$$

The value of the option is

$$e^{-0.07\times 0.25}[0.5\times 3+0.5\times 0]=1.474$$

Problem 16.23.

It is February 4. July call options on corn futures with strike prices of 260, 270, 280, 290, and 300 cost 26.75, 21.25, 17.25, 14.00, and 11.375, respectively. July put options with these strike prices cost 8.50, 13.50, 19.00, 25.625, and 32.625, respectively. The options mature on June 19, the current July corn futures price is 278.25, and the risk-free interest rate is 1.1%. Calculate implied volatilities for the options using DerivaGem. Comment on the results you get.

There are 135 days to maturity (assuming this is not a leap year). Using DerivaGem with $F_0 = 278.25$, $r = 1.1\%$, $T = 135/365$, and 500 time steps gives the implied volatilities shown in the table below.

Strike Price	Call Price	Put Price	Call Imp Vol	Put Imp Vol
260	26.75	8.50	24.69	24.59
270	21.25	13.50	25.40	26.14
280	17.25	19.00	26.85	26.86
290	14.00	25.625	28.11	27.98
300	11.375	32.625	29.24	28.57
310	9.25		34.32	

We do not expect put–call parity to hold exactly for American options and so there is no reason why the implied volatility of a call should be exactly the same as the implied volatility of a put. Nevertheless it is reassuring that they are close.

There is a tendency for high strike price options to have a higher implied volatility. As explained in Chapter 18, this is an indication that the probability distribution for corn futures prices in the future has a heavier right tail and less heavy left tail than the lognormal distribution.

Problem 16.24.

Calculate the implied volatility of soybean futures prices from the following information concerning a European put on soybean futures:

Current futures price	*525*
Exercise price	*525*
Risk-free rate	*6% per annum*
Time to maturity	*5 months*
Put price	*20*

In this case $F_0 = 525$, $K = 525$, $r = 0.06$, $T = 0.4167$. We wish to find the value of σ for which $p = 20$ where:

$$p = Ke^{-rT}N(-d_2) - F_0e^{-rT}N(-d_1)$$

This must be done by trial and error. When $\sigma = 0.2$, $p = 26.36$. When $\sigma = 0.15$, $p = 19.78$. When $\sigma = 0.155$, $p = 20.44$. When $\sigma = 0.152$, $p = 20.04$. These calculations show that the implied volatility is approximately 15.2% per annum.

Problem 16.25.

Calculate the price of a six-month European put option on the spot value of the S&P 500. The six-month forward price of the index is 1,400, the strike price is 1,450, the risk-free rate is 5%, and the volatility of the index is 15%.

The price of the option is the same as the price of a European put option on the forward price of the index where the forward contract has a maturity of six months. It is given by equation (16.10) with $F_0 = 1400$, $K = 1450$, $r = 0.05$, $\sigma = 0.15$, and $T = 0.5$. It is 86.35.

CHAPTER 17
The Greek Letters

Notes for the Instructor

This chapter covers the way in which traders working for financial institutions and market makers on the floor of an exchange hedge portfolio of derivatives. Students generally enjoy the chapter. The software, DerivaGem for Excel, can be used to demonstrate the relationships between any of the Greek letters and variables such as S_0, K, r, σ, and T.

The chapter has been restructured for the seventh edition. Up to Section 17.12 the presentation now focuses on the calculation of Greek letters for stocks. Section 17.12 then extends the results to other underlying assets (stock indices, currencies and futures). Section 17.12 also covers the difference between the delta of futures and forward contracts. This restructuring, suggested by an instructor who adopted the book, creates a significant improvement in the way the material is presented.

It is important to make sure that students understand what is meant by hedging and in particular what constitutes a good hedge. A financial institution is well hedged with respect to an underlying variable if its wealth position is largely unaffected by changes in the value of the variable. The naked positions and covered positions described in Section 17.2 are clearly not perfect hedges. The deceptively simple stop-loss rule in Section 17.3 is also far from perfect. Delta hedging works better. In fact, it works perfectly if volatility is constant and the position in the underlying asset is changed continuously. In practice of course positions cannot be changed continuously and volatility is not constant so that delta hedging is than less than perfect (See Tables 17.2, 17.3 and 17.4 for the impact of discrete rebalancing.)

I spend some time on Figure 17.7. It shows that the error in delta hedging depends on the curvature of the relationship between the derivative's price and the price of the underlying asset. This observation provides a lead in to gamma, which measures curvature. I find it worth going through a numerical example to show how a portfolio that is both gamma-neutral and delta-neutral can be constructed.

Theta is not the same type of hedge statistic as delta and gamma because there is no uncertainty about the rate at which time will pass. It is an interesting description of one aspect of a portfolio of derivatives. When delta is zero, equation (17.4) shows that theta is a proxy for gamma. When gamma is large and negative theta is large and positive, and vice versa.

Whereas gamma hedging protects the hedger against the fact that the hedge can only be adjusted discretely (e.g. every day or every week), vega hedging protects against volatility changes. Generally gamma is more important for short-life options while vega is more important for long-life options. This is clear from the equations for them. Gamma is proportional to $1/\sqrt{T}$; vega is proportional to $\sqrt{T}$.

Portfolio insurance usually generates a lively discussion—particularly if students are familiar with the details of the October 19, 1987 crash. It is important to explain that portfolio insurance involves creating a long position in an option synthetically. By contrast, hedging a long option position involves creating a short position in the option synthetically.

Problems 17.24, 17.25, 17.26,17.27 (more difficult), and 17.30 (more difficult) all make good assignment questions.

QUESTIONS AND PROBLEMS

Problem 17.1.

Explain how a stop-loss hedging scheme can be implemented for the writer of an out-of-the-money call option. Why does it provide a relatively poor hedge?

Suppose the strike price is 10.00. The option writer aims to be fully covered whenever the option is in the money and naked whenever it is out of the money. The option writer attempts to achieve this by buying the assets underlying the option as soon as the asset price reaches 10.00 from below and selling as soon as the asset price reaches 10.00 from above. The trouble with this scheme is that it assumes that when the asset price moves from 9.99 to 10.00, the next move will be to a price above 10.00. (In practice the next move might back to 9.99.) Similarly it assumes that when the asset price moves from 10.01 to 10.00, the next move will be to a price below 10.00. (In practice the next move might be back to 10.01.) The scheme can be implemented by buying at 10.01 and selling at 9.99. However, it is not a good hedge. The cost of the trading strategy is zero if the asset price never reaches 10.00 and can be quite high if it reaches 10.00 many times. A good hedge has the property that its cost is always very close the value of the option.

Problem 17.2.

What does it mean to assert that the delta of a call option is 0.7? How can a short position in 1,000 options be made delta neutral when the delta of each option is 0.7?

A delta of 0.7 means that, when the price of the stock increases by a small amount, the price of the option increases by 70% of this amount. Similarly, when the price of the stock decreases by a small amount, the price of the option decreases by 70% of this amount. A short position in 1,000 options has a delta of -700 and can be made delta neutral with the purchase of 700 shares.

Problem 17.3.

Calculate the delta of an at-the-money six-month European call option on a non-dividend-paying stock when the risk-free interest rate is 10% per annum and the stock price volatility is 25% per annum.

In this case $S_0 = K$, $r = 0.1$, $\sigma = 0.25$, and $T = 0.5$. Also,

$$d_1 = \frac{\ln(S_0/K) + (0.1 + 0.25^2/2)0.5}{0.25\sqrt{0.5}} = 0.3712$$

The delta of the option is $N(d_1)$ or 0.64.

Problem 17.4.

What does it mean to assert that the theta of an option position is –0.1 when time is measured in years? If a trader feels that neither a stock price nor its implied volatility will change, what type of option position is appropriate?

A theta of -0.1 means that if Δt units of time pass with no change in either the stock price or its volatility, the value of the option declines by $0.1\Delta t$. A trader who feels that neither the stock price nor its implied volatility will change should write an option with as high a negative theta as possible. Relatively short-life at-the-money options have the most negative thetas.

Problem 17.5.

What is meant by the gamma of an option position? What are the risks in the situation where the gamma of a position is large and negative and the delta is zero?

The gamma of an option position is the rate of change of the delta of the position with respect to the asset price. For example, a gamma of 0.1 would indicate that when the asset price increases by a certain small amount delta increases by 0.1 of this amount. When the gamma of an option writer's position is large and negative and the delta is zero, the option writer will lose significant amounts of money if there is a large movement (either an increase or a decrease) in the asset price.

Problem 17.6.

"The procedure for creating an option position synthetically is the reverse of the procedure for hedging the option position." Explain this statement.

To hedge an option position it is necessary to create the opposite option position synthetically. For example, to hedge a long position in a put it is necessary to create a short position in a put synthetically. It follows that the procedure for creating an option position synthetically is the reverse of the procedure for hedging the option position.

Problem 17.7.

Why did portfolio insurance not work well on October 19, 1987?

Portfolio insurance involves creating a put option synthetically. It assumes that as soon as a portfolio's value declines by a small amount the portfolio manager's position is rebalanced by either (a) selling part of the portfolio, or (b) selling index futures. On October 19, 1987, the market declined so quickly that the sort of rebalancing anticipated in portfolio insurance schemes could not be accomplished.

Problem 17.8.

The Black–Scholes price of an out-of-the-money call option with an exercise price of \$40 is \$4. A trader who has written the option plans to use a stop-loss strategy. The trader's plan is to buy at \$40.10 and to sell at \$39.90. Estimate the expected number of times the stock will be bought or sold.

The strategy costs the trader 0.10 each time the stock is bought or sold. The total expected cost of the strategy, in present value terms, must be \$4. This means that the expected number of times the stock will be bought or sold is approximately 40. The expected number of times it will be bought is approximately 20 and the expected number of times it will be sold is also approximately 20. The buy and sell transactions can take place at any time during the life of the option. The above numbers are therefore only approximately correct because of the effects of discounting. Also the estimate is of the number of times the stock is bought or sold in the risk-neutral world, not the real world.

Problem 17.9.

Suppose that a stock price is currently $20 and that a call option with an exercise price of $25 is created synthetically using a continually changing position in the stock. Consider the following two scenarios:

a. *Stock price increases steadily from $20 to $35 during the life of the option.*
b. *Stock price oscillates wildly, ending up at $35.*

Which scenario would make the synthetically created option more expensive? Explain your answer.

The holding of the stock at any given time must be $N(d_1)$. Hence the stock is bought just after the price has risen and sold just after the price has fallen. (This is the buy high sell low strategy referred to in the text.) In the first scenario the stock is continually bought. In second scenario the stock is bought, sold, bought again, sold again, etc. The final holding is the same in both scenarios. The buy, sell, buy, sell... situation clearly leads to higher costs than the buy, buy, buy... situation. This problem emphasizes one disadvantage of creating options synthetically. Whereas the cost of an option that is purchased is known up front and depends on the forecasted volatility, the cost of an option that is created synthetically is not known up front and depends on the volatility actually encountered.

Problem 17.10.

What is the delta of a short position in 1,000 European call options on silver futures? The options mature in eight months, and the futures contract underlying the option matures in nine months. The current nine-month futures price is $8 per ounce, the exercise price of the options is $8, the risk-free interest rate is 12% per annum, and the volatility of silver is 18% per annum.

The delta of a European futures call option is usually defined as the rate of change of the option price with respect to the futures price (not the spot price). It is

$$e^{-rT}N(d_1)$$

In this case $F_0 = 8$, $K = 8$, $r = 0.12$, $\sigma = 0.18$, $T = 0.6667$

$$d_1 = \frac{\ln(8/8) + (0.18^2/2) \times 0.6667}{0.18\sqrt{0.6667}} = 0.0735$$

$N(d_1) = 0.5293$ and the delta of the option is

$$e^{-0.12 \times 0.6667} \times 0.5293 = 0.4886$$

The delta of a short position in 1,000 futures options is therefore -488.6.

Problem 17.11.

In Problem 17.10, what initial position in nine-month silver futures is necessary for delta hedging? If silver itself is used, what is the initial position? If one-year silver futures are used, what is the initial position? Assume no storage costs for silver.

In order to answer this problem it is important to distinguish between the rate of change of the option with respect to the futures price and the rate of change of its price with respect to the spot price.

The former will be referred to as the futures delta; the latter will be referred to as the spot delta. The futures delta of a nine-month futures contract to buy one ounce of silver is by definition 1.0. Hence, from the answer to Problem 17.10, a long position in nine-month futures on 488.6 ounces is necessary to hedge the option position.

The spot delta of a nine-month futures contract is $e^{0.12\times 0.75} = 1.094$ assuming no storage costs. (This is because silver can be treated in the same way as a non-dividend-paying stock when there are no storage costs. $F_0 = S_0 e^{rT}$ so that the spot delta is the futures delta times e^{rT}) Hence the spot delta of the option position is $-488.6 \times 1.094 = -534.6$. Thus a long position in 534.6 ounces of silver is necessary to hedge the option position.

The spot delta of a one-year silver futures contract to buy one ounce of silver is $e^{0.12} = 1.1275$. Hence a long position in $e^{-0.12} \times 534.6 = 474.1$ ounces of one-year silver futures is necessary to hedge the option position.

Problem 17.12.

A company uses delta hedging to hedge a portfolio of long positions in put and call options on a currency. Which of the following would give the most favorable result?

a. A virtually constant spot rate

b. Wild movements in the spot rate

Explain your answer.

A long position in either a put or a call option has a positive gamma. From Figure 17.8, when gamma is positive the hedger gains from a large change in the stock price and loses from a small change in the stock price. Hence the hedger will fare better in case (b).

Problem 17.13.

Repeat Problem 17.12 for a financial institution with a portfolio of short positions in put and call options on a currency.

A short position in either a put or a call option has a negative gamma. From Figure 17.8, when gamma is negative the hedger gains from a small change in the stock price and loses from a large change in the stock price. Hence the hedger will fare better in case (a).

Problem 17.14.

A financial institution has just sold 1,000 seven-month European call options on the Japanese yen. Suppose that the spot exchange rate is 0.80 cent per yen, the exercise price *is 0.81 cent per yen, the risk-free interest rate in the United States is 8% per annum, the risk-free interest rate in Japan is 5% per annum, and the volatility of the yen is 15% per annum. Calculate the delta, gamma, vega, theta, and rho of the financial institution's position. Interpret each number.*

In this case $S_0 = 0.80$, $K = 0.81$, $r = 0.08$, $r_f = 0.05$, $\sigma = 0.15$, $T = 0.5833$

$$d_1 = \frac{\ln(0.80/0.81) + \left(0.08 - 0.05 + 0.15^2/2\right) \times 0.5833}{0.15\sqrt{0.5833}} = 0.1016$$
$$d_2 = d_1 - 0.15\sqrt{0.5833} = -0.0130$$

$$N(d_1) = 0.5405; \quad N(d_2) = 0.4998$$

The delta of one call option is $e^{-r_f T}N(d_1) = e^{-0.05\times0.5833} \times 0.5405 = 0.5250$.

$$N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2} = \frac{1}{\sqrt{2\pi}}e^{-0.00516} = 0.3969$$

so that the gamma of one call option is

$$\frac{N'(d_1)e^{-r_f T}}{S_0\sigma\sqrt{T}} = \frac{0.3969 \times 0.9713}{0.80 \times 0.15 \times \sqrt{0.5833}} = 4.206$$

The vega of one call option is

$$S_0\sqrt{T}N'(d_1)e^{-r_f T} = 0.80\sqrt{0.5833} \times 0.3969 \times 0.9713 = 0.2355$$

The theta of one call option is

$$\begin{aligned}
&-\frac{S_0 N'(d_1)\sigma e^{-r_f T}}{2\sqrt{T}} + r_f S_0 N(d_1)e^{-r_f T} - rKe^{-rT}N(d_2)\\
=&-\frac{0.8 \times 0.3969 \times 0.15 \times 0.9713}{2\sqrt{0.5833}}\\
&+ 0.05 \times 0.8 \times 0.5405 \times 0.9713 - 0.08 \times 0.81 \times 0.9544 \times 0.4948\\
=&-0.0399
\end{aligned}$$

The rho of one call option is

$$\begin{aligned}
&KTe^{-rT}N(d_2)\\
=&0.81 \times 0.5833 \times 0.9544 \times 0.4948\\
=&0.2231
\end{aligned}$$

Delta can be interpreted as meaning that, when the spot price increases by a small amount (measured in cents), the value of an option to buy one yen increases by 0.525 times that amount. Gamma can be interpreted as meaning that, when the spot price increases by a small amount (measured in cents), the delta increases by 4.206 times that amount. Vega can be interpreted as meaning that, when the volatility (measured in decimal form) increases by a small amount, the option's value increases by 0.2355 times that amount. When volatility increases by 1% (= 0.01) the option price increases by 0.002355. Theta can be interpreted as meaning that, when a small amount of time (measured in years) passes, the option's value decreases by 0.0399 times that amount. In particular when one calendar day passes it decreases by 0.0399/365 = 0.000109. Finally, rho can be interpreted as meaning that, when the interest rate (measured in decimal form) increases by a small amount the option's value increases by 0.2231 times that amount. When the interest rate increases by 1% (= 0.01), the options value increases by 0.002231.

Problem 17.15.

Under what circumstances is it possible to make a European option on a stock index both gamma neutral and vega neutral by adding a position in one other European option?

Assume that S_0, K, r, σ, T, q are the parameters for the option held and S_0, K^*, r, σ, T^*, q are the parameters for another option. Suppose that d_1 has its usual meaning and is calculated on the basis of the first set of parameters while d_1^* is the value of d_1 calculated on the basis of the second set of parameters. Suppose further that w of the second option are held for each of the first option held. The gamma of the portfolio is:

$$\alpha\left[\frac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}+w\frac{N'(d_1^*)e^{-qT^*}}{S_0\sigma\sqrt{T^*}}\right]$$

where α is the number of the first option held.

Since we require gamma to be zero:

$$w=-\frac{N'(d_1)e^{-q(T-T^*)}}{N'(d_1^*)}\sqrt{\frac{T^*}{T}}$$

The vega of the portfolio is:

$$\alpha\left[S_0\sqrt{T}N'(d_1)e^{-q(T)}+wS_0\sqrt{T^*}N'(d_1^*)e^{-q(T^*)}\right]$$

Since we require vega to be zero:

$$w=-\sqrt{\frac{T}{T^*}}\frac{N'(d_1)e^{-q(T-T^*)}}{N'(d_1^*)}$$

Equating the two expressions for w

$$T^*=T$$

Hence the maturity of the option held must equal the maturity of the option used for hedging.

Problem 17.16.

A fund manager has a well-diversified portfolio that mirrors the performance of the S&P 500 and is worth $360 million. The value of the S&P 500 is 1,200, and the portfolio manager would like to buy insurance against a reduction of more than 5% in the value of the portfolio over the next six months. The risk-free interest rate is 6% per annum. The dividend yield on both the portfolio and the S&P 500 is 3%, and the volatility of the index is 30% per annum.

a. *If the fund manager buys traded European put options, how much would the insurance cost?*
b. *Explain carefully alternative strategies open to the fund manager involving traded European call options, and show that they lead to the same result.*
c. *If the fund manager decides to provide insurance by keeping part of the portfolio in risk-free securities, what should the initial position be?*
d. *If the fund manager decides to provide insurance by using nine-month index futures, what should the initial position be?*

The fund is worth $300,000 times the value of the index. When the value of the portfolio falls by 5% (to $342 million), the value of the S&P 500 also falls by 5% to 1140. The fund manager therefore requires European put options on 300,000 times the S&P 500 with exercise price 1140.

(a) $S_0 = 1200$, $K = 1140$, $r = 0.06$, $\sigma = 0.30$, $T = 0.50$ and $q = 0.03$. Hence:

$$d_1 = \frac{\ln(1200/1140) + (0.06 - 0.03 + 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = 0.4186$$

$$d_2 = d_1 - 0.3\sqrt{0.5} = 0.2064$$

$$N(d_1) = 0.6622; \quad N(d_2) = 0.5818$$

$$N(-d_1) = 0.3378; \quad N(-d_2) = 0.4182$$

The value of one put option is

$$\begin{aligned}&1140e^{-rT}N(-d_2) - 1200e^{-qT}N(-d_1)\\ =&1140e^{-0.06\times0.5} \times 0.4182 - 1200e^{-0.03\times0.5} \times 0.3378\\ =&63.40\end{aligned}$$

The total cost of the insurance is therefore

$$300,000 \times 63.40 = \$19,020,000$$

(b) From put–call parity

$$S_0e^{-qT} + p = c + Ke^{-rT}$$

or:

$$p = c - S_0e^{-qT} + Ke^{-rT}$$

This shows that a put option can be created by selling (or shorting) e^{-qT} of the index, buying a call option and investing the remainder at the risk-free rate of interest. Applying this to the situation under consideration, the fund manager should:

1) Sell $360e^{-0.03\times0.5} = \$354.64$ million of stock
2) Buy call options on 300,000 times the S&P 500 with exercise price 1140 and maturity in six months.
3) Invest the remaining cash at the risk-free interest rate of 6% per annum.

This strategy gives the same result as buying put options directly.

(c) The delta of one put option is

$$\begin{aligned}&e^{-qT}[N(d_1) - 1]\\=&e^{-0.03\times0.5}(0.6622 - 1)\\-&0.3327\end{aligned}$$

This indicates that 33.27% of the portfolio (i.e., \$119.77 million) should be initially sold and invested in risk-free securities.

(d) The delta of a nine-month index futures contract is

$$e^{(r-q)T} = e^{0.03\times0.75} = 1.023$$

The spot short position required is

$$\frac{119,770,000}{1200} = 99,808$$

times the index. Hence a short position in

$$\frac{99,808}{1.023 \times 250} = 390$$

futures contracts is required.

Problem 17.17.

Repeat Problem 17.16 on the assumption that the portfolio has a beta of 1.5. Assume that the dividend yield on the portfolio is 4% per annum.

When the value of the portfolio goes down 5% in six months, the total return from the portfolio, including dividends, in the six months is

$$-5 + 2 = -3\%$$

i.e., −6% per annum. This is 12% per annum less than the risk-free interest rate. Since the portfolio has a beta of 1.5 we would expect the market to provide a return of 8% per annum less than the risk-free interest rate, i.e., we would expect the market to provide a return of −2% per annum. Since dividends on the market index are 3% per annum, we would expect the market index to have dropped at the rate of 5% per annum or 2.5% per six months; i.e., we would expect the market to have dropped to 1170. A total of $450,000 = (1.5 \times 300,000)$ put options on the S&P 500 with exercise price 1170 and exercise date in six months are therefore required.

(a) $S_0 = 1200$, $K = 1170$, $r = 0.06$, $\sigma = 0.3$, $T = 0.5$ and $q = 0.03$. Hence

$$d_1 = \frac{\ln(1200/1170) + (0.06 - 0.03 + 0.09/2) \times 0.5}{0.3\sqrt{0.5}} = 0.2961$$

$$d_2 = d_1 - 0.3\sqrt{0.5} = 0.0840$$

$$N(d_1) = 0.6164; \quad N(d_2) = 0.5335$$

$$N(-d_1) = 0.3836; \quad N(-d_2) = 0.4665$$

The value of one put option is

$$\begin{aligned} &Ke^{-rT}N(-d_2) - S_0e^{-qT}N(-d_1) \\ =&1170e^{-0.06\times0.5} \times 0.4665 - 1200e^{-0.03\times0.5} \times 0.3836 \\ =&76.28 \end{aligned}$$

The total cost of the insurance is therefore

$$450,000 \times 76.28 = \$34,326,000$$

Note that this is significantly greater than the cost of the insurance in Problem 17.16.

(b) As in Problem 17.16 the fund manager can 1) sell \$354.64 million of stock, 2) buy call options on 450,000 times the S&P 500 with exercise price 1170 and exercise date in six months and 3) invest the remaining cash at the risk-free interest rate.

(c) The portfolio is 50% more volatile than the S&P 500. When the insurance is considered as an option on the portfolio the parameters are as follows: $S_0 = 360$, $K = 342$, $r = 0.06$, $\sigma = 0.45$, $T = 0.5$ and $q = 0.04$

$$d_1 = \frac{\ln(360/342) + (0.06 - 0.04 + 0.45^2/2) \times 0.5}{0.45\sqrt{0.5}} = 0.3517$$

$$N(d_1) = 0.6374$$

The delta of the option is

$$\begin{aligned} &e^{-qT}[N(d_1) - 1] \\ =&e^{-0.04\times0.5}(0.6374 - 1) \\ =& -0.355 \end{aligned}$$

This indicates that 35.5% of the portfolio (i.e., \$127.8 million) should be sold and invested in riskless securities.

(d) We now return to the situation considered in (a) where put options on the index are required. The delta of each put option is

$$\begin{aligned}
&e^{-qT}(N(d_1)-1)\\
=&e^{-0.03\times0.5}(0.6164-1)\\
=&-0.3779
\end{aligned}$$

The delta of the total position required in put options is $-450,000 \times 0.3779 = -170,000$. The delta of a nine month index futures is (see Problem 17.16) 1.023. Hence a short position in

$$\frac{170,000}{1.023 \times 250} = 665$$

index futures contracts.

Problem 17.18.

Show by substituting for the various terms in equation (17.4) that the equation is true for:

a. *A single European call option on a non-dividend-paying stock*
b. *A single European put option on a non-dividend-paying stock*
c. *Any portfolio of European put and call options on a non-dividend-paying stock*

(a) For a call option on a non-dividend-paying stock

$$\begin{aligned}
\Delta &= N(d_1)\\
\Gamma &= \frac{N'(d_1)}{S_0\sigma\sqrt{T}}\\
\Theta &= -\frac{S_0N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)
\end{aligned}$$

Hence the left-hand side of equation (17.4) is:

$$\begin{aligned}
=&-\frac{S_0N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) + rS_0N(d_1) + \frac{1}{2}\sigma S_0\frac{N'(d_1)}{\sqrt{T}}\\
=&r[S_0N(d_1) - Ke^{-rT}N(d_2)]\\
=&r\Pi
\end{aligned}$$

(b) For a put option on a non-dividend-paying stock

$$\begin{aligned}
\Delta &= N(d_1) - 1 = -N(-d_1)\\
\Gamma &= \frac{N'(d_1)}{S_0\sigma\sqrt{T}}\\
\Theta &= -\frac{S_0N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)
\end{aligned}$$

Hence the left-hand side of equation (17.4) is:

$$
\begin{aligned}
&-\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2) - rS_0N(-d_1) + \frac{1}{2}\sigma S_0\frac{N'(d_1)}{\sqrt{T}}\\
&=r[Ke^{-rT}N(-d_2) - S_0N(-d_1)]\\
&=r\Pi
\end{aligned}
$$

(c) For a portfolio of options, Π, Δ, Θ and Γ are the sums of their values for the individual options in the portfolio. It follows that equation (17.4) is true for any portfolio of European put and call options.

Problem 17.19

What is the equation corresponding to equation (17.4) for (a) a portfolio of derivatives on a currency and (b) a portfolio of derivatives on a futures contract?

A currency is analogous to a stock paying a continuous dividend yield at rate r_f. The differential equation for a portfolio of derivatives dependent on a currency is (see equation 15.6)

$$\frac{\partial \Pi}{\partial t} + (r - r_f)S\frac{\partial \Pi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2} = r\Pi$$

Hence

$$\Theta + (r - r_f)S\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

Similarly, for a portfolio of derivatives dependent on a futures price (see equation 16.8)

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

Problem 17.20.

Suppose that \$70 billion of equity assets are the subject of portfolio insurance schemes. Assume that the schemes are designed to provide insurance against the value of the assets declining by more than 5% within one year. Making whatever estimates you find necessary, use the DerivaGem software to calculate the value of the stock or futures contracts that the administrators of the portfolio insurance schemes will attempt to sell if the market falls by 23% in a single day.

We can regard the position of all portfolio insurers taken together as a single put option. The three known parameters of the option, before the 23% decline, are $S_0 = 70$, $K = 66.5$, $T = 1$. Other parameters can be estimated as $r = 0.06$, $\sigma = 0.25$ and $q = 0.03$. Then:

$$d_1 = \frac{\ln(70/66.5) + (0.06 - 0.03 + 0.25^2/2)}{0.25} = 0.4502$$

$$N(d_1) = 0.6737$$

The delta of the option is

$$\begin{aligned}&e^{-qT}[N(d_1) - 1]\\=&e^{-0.03}(0.6737 - 1)\\=&-0.3167\end{aligned}$$

This shows that 31.67% or \$22.17 billion of assets should have been sold before the decline. These numbers can also be produced from DerivaGem by selecting Underlying Type and Index and Option Type as Analytic European.

After the decline, $S_0 = 53.9$, $K = 66.5$, $T = 1$, $r = 0.06$, $\sigma = 0.25$ and $q = 0.03$.

$$d_1 = \frac{\ln(53.9/66.5) + (0.06 - 0.03 + 0.25^2/2)}{0.25} = -0.5953$$

$$N(d_1) = 0.2758$$

The delta of the option has dropped to

$$\begin{aligned}&e^{-0.03\times0.5}(0.2758 - 1)\\=&-0.7028\end{aligned}$$

This shows that cumulatively 70.28% of the assets originally held should be sold. An additional 38.61% of the original portfolio should be sold. The sales measured at pre-crash prices are about \$27.0 billion. At post crash prices they are about 20.8 billion.

Problem 17.21.

Does a forward contract on a stock index have the same delta as the corresponding futures contract? Explain your answer.

With our usual notation the value of a forward contract on the asset is $S_0e^{-qT} - Ke^{-rT}$. When there is a small change, ΔS, in S_0 the value of the forward contract changes by $e^{-qT}\Delta S$. The delta of the forward contract is therefore e^{-qT}. The futures price is $S_0e^{(r-q)T}$. When there is a small change, ΔS, in S_0 the futures price changes by $\Delta Se^{(r-q)T}$. Given the daily settlement procedures in futures contracts, this is also the immediate change in the wealth of the holder of the futures contract. The delta of the futures contract is therefore $e^{(r-q)T}$. We conclude that the deltas of a futures and forward contract are not the same. The delta of the futures is greater than the delta of the corresponding forward by a factor of e^{rT}.

Problem 17.22.

A bank's position in options on the dollar–euro exchange rate has a delta of 30,000 and a gamma of $-80,000$. Explain how these numbers can be interpreted. The exchange rate (dollars per euro) is 0.90. What position would you take to make the position delta neutral? After a short period of time, the exchange rate moves to 0.93. Estimate the new delta. What additional trade is necessary to keep the position delta neutral? Assuming the bank did set up a delta-neutral position originally, has it gained or lost money from the exchange-rate movement?

The delta indicates that when the value of the euro exchange rate increases by \$0.01, the value of the bank's position increases by $0.01 \times 30,000 = \$300$. The gamma indicates that when the euro exchange rate increases by \$0.01 the delta of the portfolio decreases by $0.01 \times 80,000 = 800$. For delta neutrality 30,000 euros should be shorted. When the exchange rate moves up to 0.93, we expect the delta of the portfolio to decrease by $(0.93 - 0.90) \times 80,000 = 2,400$ so that it becomes 27,600. To maintain delta neutrality, it is therefore necessary for the bank to unwind its short position 2,400 euros so that a net 27,600 have been shorted. As shown in the text (see Figure 17.8), when a portfolio is delta neutral and has a negative gamma, a loss is experienced when there is a large movement in the underlying asset price. We can conclude that the bank is likely to have lost money.

Problem 17.23.

Use the put–call parity relationship to derive, for a non-dividend-paying stock, the relationship between:

(a) The delta of a European call and the delta of a European put.
(b) The gamma of a European call and the gamma of a European put.
(c) The vega of a European call and the vega of a European put.
(d) The theta of a European call and the theta of a European put.

For a non-dividend paying stock, put-call parity gives at a general time t:

$$p + S = c + Ke^{-r(T-t)}$$

(a) Differentiating with respect to S:

$$\frac{\partial p}{\partial S} + 1 = \frac{\partial c}{\partial S}$$

or

$$\frac{\partial p}{\partial S} = \frac{\partial c}{\partial S} - 1$$

This shows that the delta of a European put equals the delta of the corresponding European call less 1.0.

(b) Differentiating with respect to S again

$$\frac{\partial^2 p}{\partial S^2} = \frac{\partial^2 c}{\partial S^2}$$

Hence the gamma of a European put equals the gamma of a European call.

(c) Differentiating the put-call parity relationship with respect to σ

$$\frac{\partial p}{\partial \sigma} = \frac{\partial c}{\partial \sigma}$$

showing that the vega of a European put equals the vega of a European call.

(d) Differentiating the put-call parity relationship with respect to T

$$\frac{\partial p}{\partial t} = rKe^{-r(T-t)} + \frac{\partial c}{\partial t}$$

This is in agreement with the thetas of European calls and puts given in Section 17.5 since $N(d_2) = 1 - N(-d_2)$.

ASSIGNMENT QUESTIONS

Problem 17.24.

Consider a one-year European call option on a stock when the stock price is \$30, the strike price is \$30, the risk-free rate is 5%, and the volatility is 25% per annum. Use the DerivaGem software to calculate the price, delta, gamma, vega, theta, and rho of the option. Verify that delta is correct by changing the stock price to \$30.1 and recomputing the option price. Verify that gamma is correct by recomputing the delta for the situation where the stock price is \$30.1. Carry out similar calculations to verify that vega, theta, and rho are correct. Use the DerivaGem software to plot the option price, delta, gamma, vega, theta, and rho against the stock price for the stock option.

The price, delta, gamma, vega, theta, and rho of the option are 3.7008, 0.6274, 0.050, 0.1135, -0.00596, and 0.1512. When the stock price increases to 30.1, the option price increases to 3.7638. The change in the option price is $3.7638 - 3.7008 = 0.0630$. Delta predicts a change in the option price of $0.6274 \times 0.1 = 0.0627$ which is very close. When the stock price increases to 30.1, delta increases to 0.6324. The size of the increase in delta is $0.6324 - 0.6274 = 0.005$. Gamma predicts an increase of $0.050 \times 0.1 = 0.005$ which is the same. When the volatility increases from 25% to 26%, the option price increases by 0.1136 from 3.7008 to 3.8144. This is consistent with the vega value of 0.1135. When the time to maturity is changed from 1 to 1-1/365 the option price reduces by 0.006 from 3.7008 to 3.6948. This is consistent with a theta of -0.00596. Finally when the interest rate increases from 5% to 6% the value of the option increases by 0.1527 from 3.7008 to 3.8535. This is consistent with a rho of 0.1512.

Problem 17.25.

A financial institution has the following portfolio of over-the-counter options on sterling:

Type	*Position*	*Delta of Option*	*Gamma of Option*	*Vega of Option*
Call	*–1,000*	*0.50*	*2.2*	*1.8*
Call	*–500*	*0.80*	*0.6*	*0.2*
Put	*–2,000*	*–0.40*	*1.3*	*0.7*
Call	*–500*	*0.70*	*1.8*	*1.4*

A traded option is available with a delta of 0.6, a gamma of 1.5, and a vega of 0.8.

a. *What position in the traded option and in sterling would make the portfolio both gamma neutral and delta neutral?*

b. What position in the traded option and in sterling would make the portfolio both vega neutral and delta neutral?

The delta of the portfolio is

$$-1,000 \times 0.50 - 500 \times 0.80 - 2,000 \times (-0.40) - 500 \times 0.70 = -450$$

The gamma of the portfolio is

$$-1,000 \times 2.2 - 500 \times 0.6 - 2,000 \times 1.3 - 500 \times 1.8 = -6,000$$

The vega of the portfolio is

$$-1,000 \times 1.8 - 500 \times 0.2 - 2,000 \times 0.7 - 500 \times 1.4 = -4,000$$

(a) A long position in 4,000 traded options will give a gamma-neutral portfolio since the long position has a gamma of $4,000 \times 1.5 = +6,000$. The delta of the whole portfolio (including traded options) is then:

$$4,000 \times 0.6 - 450 = 1,950$$

Hence, in addition to the 4,000 traded options, a short position in £1,950 is necessary so that the portfolio is both gamma and delta neutral.

(b) A long position in 5,000 traded options will give a vega-neutral portfolio since the long position has a vega of $5,000 \times 0.8 = +4,000$. The delta of the whole portfolio (including traded options) is then

$$5,000 \times 0.6 - 450 = 2,550$$

Hence, in addition to the 5,000 traded options, a short position in £2,550 is necessary so that the portfolio is both vega and delta neutral.

Problem 17.26.

Consider again the situation in Problem 17.25. Suppose that a second traded option with a delta of 0.1, a gamma of 0.5, and a vega of 0.6 is available. How could the portfolio be made delta, gamma, and vega neutral?

Let w_1 be the position in the first traded option and w_2 be the position in the second traded option. We require:

$$6,000 = 1.5w_1 + 0.5w_2$$

$$4,000 = 0.8w_1 + 0.6w_2$$

The solution to these equations can easily be seen to be $w_1 = 3,200$, $w_2 = 2,400$. The whole portfolio then has a delta of

$$-450 + 3,200 \times 0.6 + 2,400 \times 0.1 = 1,710$$

Therefore the portfolio can be made delta, gamma and vega neutral by taking a long position in 3,200 of the first traded option, a long position in 2,400 of the second traded option and a short position in £1,710.

Problem 17.27.

A deposit instrument offered by a bank guarantees that investors will receive a return during a six-month period that is the greater of (a) zero and (b) 40% of the return provided by a market index. An investor is planning to put $100,000 in the instrument. Describe the payoff as an option on the index. Assuming that the risk-free rate of interest is 8% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 25% per annum, is the product a good deal for the investor?

The product provides a six-month return equal to

$$\max(0,\ 0.4R)$$

where R is the return on the index. Suppose that S_0 is the current value of the index and S_T is the value in six months.

When an amount A is invested, the return received at the end of six months is:

$$\begin{aligned}&A\max(0,\ 0.4\frac{S_T-S_0}{S_0})\\&=\frac{0.4A}{S_0}\max(0,\ S_T-S_0)\end{aligned}$$

This is $0.4A/S_0$ of at-the-money European call options on the index. With the usual notation, they have value:

$$\begin{aligned}&\frac{0.4A}{S_0}[S_0e^{-qT}N(d_1)-S_0e^{-rT}N(d_2)]\\&=0.4A[e^{-qT}N(d_1)-e^{-rT}N(d_2)]\end{aligned}$$

In this case $r=0.08$, $\sigma=0.25$, $T=0.50$ and $q=0.03$

$$\begin{aligned}d_1&=\frac{(0.08-0.03+0.25^2/2)\,0.50}{0.25\sqrt{0.50}}=0.2298\\d_2&=d_1-0.25\sqrt{0.50}=0.0530\end{aligned}$$

$$N(d_1)=0.5909;\quad N(d_2)=0.5212$$

The value of the European call options being offered is

$$\begin{aligned}&0.4A(e^{-0.03\times0.5}\times0.5909-e^{-0.08\times0.5}\times0.5212)\\&=0.0325A\end{aligned}$$

This is the present value of the payoff from the product. If an investor buys the product he or she avoids having to pay $0.0325A$ at time zero for the underlying option. The cash flows to the investor are therefore

Time 0: $-A + 0.0325A = 0.9675A$

After six months: $+A$

The return with continuous compounding is $2\ln(1/0.9675) = 0.066$ or 6.6% per annum. The product is therefore slightly less attractive than a risk-free investment.

Problem 17.28.

The formula for the price of a European call futures option in terms of the futures price, F_0, is given in Chapter 16 as

$$c = e^{-rT}[F_0 N(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

and K, r, T, and σ are the strike price, interest rate, time to maturity, and volatility, respectively.

(a) Prove that $F_0 N'(d_1) = KN'(d_2)$

(b) Prove that the delta of the call price with respect to the futures price is $e^{-rT}N(d_1)$.

(c) Prove that the vega of the call price is $F_0\sqrt{T}N'(d_1)e^{-rT}$

(d) Prove the formula for the rho of a call futures option given in Section 17.12. The delta, gamma, theta, and vega of a call futures option are the same as those for a call option on a stock paying dividends at rate q with q replaced by r and S_0 replaced by F_0. Explain why the same is not true of the rho of a call futures option.

(a)

$$FN'(d_1) = \frac{F}{\sqrt{2\pi}}e^{-d_1^2/2}$$

$$KN'(d_2) = KN'(d_1 - \sigma\sqrt{T}) = \frac{K}{\sqrt{2\pi}}e^{-(d_1^2/2)+d_1\sigma\sqrt{T}-\sigma^2T/2}$$

Because $d_1\sigma\sqrt{T} = \ln(F/K) + \sigma^2T/2$ the second equation reduces to

$$KN'(d_2) = \frac{K}{\sqrt{2\pi}}e^{-(d_1^2/2)+ln(F/K)} = \frac{F}{\sqrt{2\pi}}e^{-d_1^2/2}$$

The result follows.

(b)

$$\frac{\partial c}{\partial F} = e^{-rT}N(d_1) + e^{-rT}FN'(d_1)\frac{\partial d_1}{\partial F} - e^{-rT}KN'(d_2)\frac{\partial d_2}{\partial F}$$

Because

$$\frac{\partial d_1}{\partial F} = \frac{\partial d_2}{\partial F}$$

it follows from the result in (a) that

$$\frac{\partial c}{\partial F} = e^{-rT}N(d_1)$$

(c)

$$\frac{\partial c}{\partial \sigma} = e^{-rT}FN'(d_1)\frac{\partial d_1}{\partial \sigma} - e^{-rT}KN'(d_2)\frac{\partial d_2}{\partial \sigma}$$

Because $d_1 = d_2 + \sigma\sqrt{T}$

$$\frac{\partial d_1}{\partial \sigma} = \frac{\partial d_2}{\partial \sigma} + \sqrt{T}$$

From the result in (a) it follows that

$$\frac{\partial c}{\partial \sigma} = e^{-rT}FN'(d_1)\sqrt{T}$$

(d) Rho is given by

$$\frac{\partial c}{\partial r} = -Te^{-rT}[FN(d_1) - KN(d_2)]]$$

or $-cT$. Because $q = r$ in the case of a futures option there are two components to rho. One arises from differentiation with respect to r, the other from differentiation with respect to q.

Problem 17.29.

Use DerivaGem to check that equation (17.4) is satisfied for the option considered in Section 17.1. (Note: DerivaGem produces a value of theta "per calendar day." The theta in equation (17.7) is "per year.")

For the option considered in Section 17.1, $S_0 = 49$, $K = 50$, $r = 0.05$, $\sigma = 0.20$, and $T = 20/52$. DerivaGem shows that $\Theta = -0.011795 \times 365 = -4.305$, $\Delta = 0.5216$, $\Gamma = 0.065544$, $\Pi = 2.4005$. The left hand side of equation (17.7)

$$-4.305 + 0.05 \times 49 \times 0.5216 + \frac{1}{2} \times 0.2^2 \times 49^2 \times 0.065544 = 0.120$$

The right hand side is

$$0.05 \times 2.4005 = 0.120$$

This shows that the result in equation (17.4) is satisfied.

Problem 17.30.

Use the DerivaGem Application Builder functions to reproduce Table 17.2. (Note that in Table 17.2 the stock position is rounded to the nearest 100 shares.) Calculate the gamma and theta of the position each week. Calculate the change in the value of the portfolio each week and check whether equation (17.3) is approximately satisfied.(Note: DerivaGem produces a value of theta "per calendar day." The theta in equation (17.3) is "per year.")

Consider the first week. The portfolio consists of a short position in 100,000 options and a long position in 52,200 shares. The value of the option changes from \$240,053 at the beginning of the week to \$188,760 at the end of the week for a gain of \$51,293. The value of the shares change from $52,200 \times 49 = \$2,557,800$ to $52,200 \times 48.12 = \$2,511,864$ for a loss of \$45,936. The net gain is $51,293 - 45,936 = \$5,357$. The gamma and theta (per year) of the portfolio are -6554.4 and 430,533 so that equation (17.3) predicts the gain as

$$430533 \times \frac{1}{52} - \frac{1}{2} \times 6554.4 \times (48.12 - 49)^2 = 5742$$

The results for all 20 weeks are shown in the following table.

Week	Actual Gain	Predicted Gain
1	5,357	5,742
2	5,689	6,093
3	-19,742	-21,084
4	1,941	1,572
5	3,706	3,652
6	9,320	9,191
7	6,249	5,936
8	9,491	9,259
9	961	870
10	-23,380	-18,992
11	1,643	2,497
12	2,645	1,356
13	11,365	10,923
14	-2,876	-3,342
15	12,936	12,302
16	7,566	8,815
17	-3,880	-2,763
18	6,764	6,899
19	4,295	5,205
20	4,804	4,805

CHAPTER 18
Volatility Smiles

Notes for the Instructor

This chapter covers volatility smiles and how they are used in practice. The approach is to start with the volatility smiles that are observed in the equity and foreign currency markets and then show what the implied distributions look like. A number of improvements have been made to the chapter. Section 18.1 reads more easily. Different ways used by practitioners to quantify the volatility smile are covered.

I find that many students are interested in the details of how one goes from a volatility smile to an implied distribution. The appendix to the chapter now has more information on this and includes a numerical example.

I focus on foreign exchange and equity markets when covering this chapter, but futures markets can also be mentioned. (Problem 16.23 from Chapter 16 derives a volatility for corn futures.)

It is not difficult to construct interesting assignments based on the material. For example, students can be asked to calculate a volatility smile for options on the S&P 500 using data obtained from a newspaper or a live data feed. Problems 18.19 and 18.26 work well for class discussion. The others make good assignment questions.

QUESTIONS AND PROBLEMS

Problem 18.1.

What volatility smile is likely to be observed when

a. *Both tails of the stock price distribution are less heavy than those of the lognormal distribution?*

b. *The right tail is heavier, and the left tail is less heavy, than that of a lognormal distribution?*

A downward sloping volatility smile is usually observed for equities.

Problem 18.2.

What volatility smile is observed for equities?

A downward sloping volatility smile is usually observed for equities.

Problem 18.3.

What volatility smile is likely to be caused by jumps in the underlying asset price? Is the pattern likely to be more pronounced for a two-year option than for a three-month option?

Jumps tend to make both tails of the stock price distribution heavier than those of the lognormal distribution. This creates a volatility smile similar to that in Figure 18.1. The volatility smile is likely to be more pronounced for the three-month option.

Problem 18.4.

A European call and put option have the same strike price and time to maturity. The call has an implied volatility of 30% and the put has an implied volatility of 25%. What trades would you do?

The put has a price that is too low relative to the call's price. The correct trading strategy is to buy the put, buy the stock, and sell the call.

Problem 18.5.

Explain carefully why a distribution with a heavier left tail and less heavy right tail than the lognormal distribution gives rise to a downward sloping volatility smile.

The heavier left tail should lead to high prices, and therefore high implied volatilities, for out-of-the-money (low-strike-price) puts. Similarly the less heavy right tail should lead to low prices, and therefore low volatilities for out-of-the-money (high-strike-price) calls. A volatility smile where volatility is a decreasing function of strike price results.

Problem 18.6.

The market price of a European call is $3.00 and its price given by Black–Scholes model with a volatility of 30% is $3.50. The price given by this Black–Scholes model for a European put option with the same strike price and time to maturity is $1.00. What should the market price of the put option be? Explain the reasons for your answer.

With the notation in the text

$$c_{\text{bs}} + Ke^{-rT} = p_{\text{bs}} + Se^{-qT}$$

$$c_{\text{mkt}} + Ke^{-rT} = p_{\text{mkt}} + Se^{-qT}$$

It follows that

$$c_{\text{bs}} - c_{\text{mkt}} = p_{\text{bs}} - p_{\text{mkt}}$$

In this case $c_{\text{mkt}} = 3.00$; $c_{\text{bs}} = 3.50$; and $p_{\text{bs}} = 1.00$. It follows that p_{mkt} should be 0.50.

Problem 18.7.

Explain what is meant by crashophobia.

The crashophobia argument is an attempt to explain the pronounced volatility skew in equity markets since 1987. (This was the year equity markets shocked everyone by crashing more than 20% in one day). The argument is that traders are concerned about another crash and as a result increase the price of out-of-the-money puts. This creates the volatility skew.

Problem 18.8.

A stock price is currently $20. Tomorrow, news is expected to be announced that will either increase the price by $5 or decrease the price by $5. What are the problems in using Black–Scholes to value one-month options on the stock?

The probability distribution of the stock price in one month is not lognormal. Possibly it consists of two lognormal distributions superimposed upon each other and is bimodal. Black–Scholes is clearly inappropriate, because it assumes that the stock price at any future time is lognormal.

Problem 18.9.

What volatility smile is likely to be observed for six-month options when the volatility is uncertain and positively correlated to the stock price?

When the asset price is positively correlated with volatility, the volatility tends to increase as the asset price increases, producing less heavy left tails and heavier right tails. Implied volatility then increases with the strike price.

Problem 18.10.

What problems do you think would be encountered in testing a stock option pricing model empirically?

There are a number of problems in testing an option pricing model empirically. These include the problem of obtaining synchronous data on stock prices and option prices, the problem of estimating the dividends that will be paid on the stock during the option's life, the problem of distinguishing between situations where the market is inefficient and situations where the option pricing model is incorrect, and the problems of estimating stock price volatility.

Problem 18.11.

Suppose that a central bank's policy is to allow an exchange rate to fluctuate between 0.97 and 1.03. What pattern of implied volatilities for options on the exchange rate would you expect to see?

In this case the probability distribution of the exchange rate has a thin left tail and a thin right tail relative to the lognormal distribution. We are in the opposite situation to that described for foreign currencies in Section 18.1. Both out-of-the-money and in-the-money calls and puts can be expected to have lower implied volatilities than at-the-money calls and puts. The pattern of implied volatilities is likely to be similar to Figure 18.7.

Problem 18.12.

Option traders sometimes refer to deep-out-of-the-money options as being options on volatility. Why do you think they do this?

A deep-out-of-the-money option has a low value. Decreases in its volatility reduce its value. However, this reduction is small because the value can never go below zero. Increases in its volatility, on the other hand, can lead to significant percentage increases in the value of the option. The option does, therefore, have some of the same attributes as an option on volatility.

Problem 18.13.

A European call option on a certain stock has a strike price of \$30, a time to maturity of one year, and an implied volatility of 30%. A European put option on the same stock has a strike price of \$30, a time to maturity of one year, and an implied volatility of 33%. What is the arbitrage opportunity open to a trader? Does the arbitrage work only when the lognormal assumption underlying Black–Scholes holds? Explain the reasons for your answer carefully.

As explained in the appendix to the chapter, put–call parity implies that European put and call options have the same implied volatility. If a call option has an implied volatility of 30% and a put option has an implied volatility of 33%, the call is priced too low relative to the put. The correct trading strategy is to buy the call, sell the put and short the stock. This does not depend on the lognormal assumption underlying Black–Scholes. Put–call parity is true for any set of assumptions.

Problem 18.14.

Suppose that the result of a major lawsuit affecting a company is due to be announced tomorrow. The company's stock price is currently \$60. If the ruling is favorable to the company, the stock price is expected to jump to \$75. If it is unfavorable, the stock is expected to jump to \$50. What is the risk-neutral probability of a favorable ruling? Assume that the volatility of the company's stock will be 25% for six months after the ruling if the ruling is favorable and 40% if it is unfavorable. Use DerivaGem to calculate the relationship between implied volatility and strike price for six-month European options on the company today. The company does not pay dividends. Assume that the six-month risk-free rate is 6%. Consider call options with strike prices of \$30, \$40, \$50, \$60, \$70, and \$80.

Suppose that p is the probability of a favorable ruling. The expected price of the company's stock tomorrow is

$$75p + 50(1 - p) = 50 + 25p$$

This must be the price of the stock today. (We ignore the expected return to an investor over one day.) Hence

$$50 + 25p = 60$$

or $p = 0.4$.

If the ruling is favorable, the volatility, σ, will be 25%. Other option parameters are $S_0 = 75$, $r = 0.06$, and $T = 0.5$. For a value of K equal to 50, DerivaGem gives the value of a European call option price as 26.502.

If the ruling is unfavorable, the volatility, σ will be 40% Other option parameters are $S_0 = 50$, $r = 0.06$, and $T = 0.5$. For a value of K equal to 50, DerivaGem gives the value of a European call option price as 6.310.

The value today of a European call option with a strike price today is the weighted average of 26.502 and 6.310 or:

$$0.4 \times 26.502 + 0.6 \times 6.310 = 14.387$$

DerivaGem can be used to calculate the implied volatility when the option has this price. The parameter values are $S_0 = 60$, $K = 50$, $T = 0.5$, $r = 0.06$ and $c = 14.387$. The implied volatility is 47.76%.

These calculations can be repeated for other strike prices. The results are shown in the table below. The pattern of implied volatilities is shown in Figure S18.1.

Strike Price	Call Option Price Favorable Outcome	Call Option Price Unfavorable Outcome	Weighted Price	Implied Volatility (%)
30	45.887	21.001	30.955	46.67
40	36.182	12.437	21.935	47.78
50	26.502	6.310	14.387	47.76
60	17.171	2.826	8.564	46.05
70	9.334	1.161	4.430	43.22
80	4.159	0.451	1.934	40.36

Figure S18.1 Implied Volatilities in Problem 18.14

Problem 18.15.

An exchange rate is currently 0.8000. The volatility of the exchange rate is quoted as 12% and interest rates in the two countries are the same. Using the lognormal assumption, estimate the probability that the exchange rate in three months will be (a) less than 0.7000, (b) between 0.7000 and 0.7500, (c) between 0.7500 and 0.8000, (d) between 0.8000 and 0.8500, (e) between 0.8500 and 0.9000, and (f) greater than 0.9000. Based on the volatility smile usually observed in the market for exchange rates, which of these estimates would you expect to be too low and which would you expect to be too high?

As pointed out in Chapters 5 and 13 an exchange rate behaves like a stock that provides a dividend yield equal to the foreign risk-free rate. Whereas the growth rate in a non-dividend-paying stock in a risk-neutral world is r, the growth rate in the exchange rate in a risk-neutral world is $r - r_f$. Exchange rates have low systematic risks and so we can reasonably assume that this is also the growth rate in the real world. In this case the foreign risk-free rate equals the domestic risk-free rate ($r = r_f$). The expected growth rate in the exchange rate is therefore zero. If S_T is the exchange rate at time T its probability distribution is given by equation (12.2) with $\mu = 0$:

$$\ln S_T \sim \phi(\ln S_0 - \sigma^2 T/2, \sigma^2 T)$$

where S_0 is the exchange rate at time zero and σ is the volatility of the exchange rate. In this case $S_0 = 0.8000$ and $\sigma = 0.12$, and $T = 0.25$ so that

$$\ln S_T \sim \phi(\ln 0.8 - 0.12^2 \times 0.25/2, 0.12^2 \times 0.25)$$

or

$$\ln S_T \sim \phi(-0.2249, 0.0036)$$

(a) $\ln 0.70 = -0.3567$. The probability that $S_T < 0.70$ is the same as the probability that $\ln S_T < -0.3567$. It is

$$N\left(\frac{-0.3567 + 0.2249}{0.06}\right) = N(-2.1955)$$

This is 1.41%.

(b) $\ln 0.75 = -0.2877$. The probability that $S_T < 0.75$ is the same as the probability that $\ln S_T < -0.2877$. It is

$$N\left(\frac{-0.2877 + 0.2249}{0.06}\right) = N(-1.0456)$$

This is 14.79%. The probability that the exchange rate is between 0.70 and 0.75 is therefore $14.79 - 1.41 = 13.38$%.

(c) $\ln 0.80 = -0.2231$. The probability that $S_T < 0.80$ is the same as the probability that $\ln S_T < -0.2231$. It is

$$N\left(\frac{-0.2231 + 0.2249}{0.06}\right) = N(0.0300)$$

This is 51.20%. The probability that the exchange rate is between 0.75 and 0.80 is therefore $51.20 - 14.79 = 36.41\%$.

(d) $\ln 0.85 = -0.1625$. The probability that $S_T < 0.85$ is the same as the probability that $\ln S_T < -0.1625$. It is

$$N\left(\frac{-0.1625 + 0.2249}{0.06}\right) = N(1.0404)$$

This is 85.09%. The probability that the exchange rate is between 0.80 and 0.85 is therefore $85.09 - 51.20 = 33.89\%$.

(e) $\ln 0.90 = -0.1054$. The probability that $S_T < 0.90$ is the same as the probability that $\ln S_T < -0.1054$. It is

$$N\left(\frac{-0.1054 + 0.2249}{0.06}\right) = N(1.9931)$$

This is 97.69%. The probability that the exchange rate is between 0.85 and 0.90 is therefore $97.69 - 85.09 = 12.60\%$.

(f) The probability that the exchange rate is greater than 0.90 is $100 - 97.69 = 2.31\%$

The volatility smile encountered for foreign exchange options is shown in Figure 18.1 of the text and implies the probability distribution in Figure 18.2. Figure 18.2 suggests that we would expect the probabilities in (a), (c), (d), and (f) to be too low and the probabilities in (b) and (e) to be too high.

Problem 18.16.

The price of a stock is $40. A six-month European call option on the stock with a strike price of $30 has an implied volatility of 35%. A six month European call option on the stock with a strike price of $50 has an implied volatility of 28%. The six-month risk-free rate is 5% and no dividends are expected. Explain why the two implied volatilities are different. Use DerivaGem to calculate the prices of the two options. Use put–call parity to calculate the prices of six-month European put options with strike prices of $30 and $50. Use DerivaGem to calculate the implied volatilities of these two put options.

The difference between the two implied volatilities is consistent with Figure 18.3 in the text. For equities the volatility smile is downward sloping. A high strike price option has a lower implied volatility than a low strike price option. The reason is that traders consider that the probability of a large downward movement in the stock price is higher than that predicted by the lognormal probability distribution. The implied distribution assumed by traders is shown in Figure 18.4.

To use DerivaGem to calculate the price of the first option, proceed as follows. Select Equity as the Underlying Type in the first worksheet. Select Analytic European as the Option Type. Input the stock price as 40, volatility as 35%, risk-free rate as 5%, time to exercise as 0.5 year, and exercise price as 30. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Do not select the implied volatility button. Hit the *Enter* key and click on calculate. DerivaGem will show the price of the option as 11.155. Change the volatility to 28% and the strike price to 50. Hit the *Enter* key and click on calculate. DerivaGem will show the price of the option as 0.725.

Put–call parity is

$$c + Ke^{-rT} = p + S_0$$

so that

$$p = c + Ke^{-rT} - S_0$$

For the first option, $c = 11.155$, $S_0 = 40$, $r = 0.054$, $K = 30$, and $T = 0.5$ so that

$$p = 11.155 + 30e^{-0.05\times0.5} - 40 = 0.414$$

For the second option, $c = 0.725$, $S_0 = 40$, $r = 0.06$, $K = 50$, and $T = 0.5$ so that

$$p = 0.725 + 50e^{-0.05\times0.5} - 40 = 9.490$$

To use DerivaGem to calculate the implied volatility of the first put option, input the stock price as 40, the risk-free rate as 5%, time to exercise as 0.5 year, and the exercise price as 30. Input the price as 0.414 in the second half of the Option Data table. Select the buttons for a put option and implied volatility. Hit the *Enter* key and click on calculate. DerivaGem will show the implied volatility as 34.99%.

Similarly, to use DerivaGem to calculate the implied volatility of the first put option, input the stock price as 40, the risk-free rate as 5%, time to exercise as 0.5 year, and the exercise price as 50. Input the price as 9.490 in the second half of the Option Data table. Select the buttons for a put option and implied volatility. Hit the *Enter* key and click on calculate. DerivaGem will show the implied volatility as 27.99%.

These results are what we would expect. DerivaGem gives the implied volatility of a put with strike price 30 to be almost exactly the same as the implied volatility of a call with a strike price of 30. Similarly, it gives the implied volatility of a put with strike price 50 to be almost exactly the same as the implied volatility of a call with a strike price of 50.

Problem 18.17.

"The Black–Scholes model is used by traders as an interpolation tool." Discuss this view.

When plain vanilla call and put options are being priced, traders do use the Black–Scholes model as an interpolation tool. They calculate implied volatilities for the options whose prices they can observe in the market. By interpolating between strike prices and between times to maturity, they estimate implied volatilities for other options. These implied volatilities are then substituted into Black–Scholes to calculate prices for these options. In practice much of the work in producing a table such as Table 18.2 in the over-the-counter market is done by brokers. Brokers often act as intermediaries between participants in the over-the-counter market and usually have more information on the trades taking place than any individual financial institution. The brokers provide a table such as Table 18.2 to their clients as a service.

Problem 18.18

Using Table 18.2 calculate the implied volatility a trader would use for an 8-month option with $K/S_0 = 1.04$.

13.45%. We get the same answer by (a) interpolating between strike prices of 1.00 and 1.05 and then between maturities six months and one year and (b) interpolating between maturities of six months and one year and then between strike prices of 1.00 and 1.05.

ASSIGNMENT QUESTIONS

Problem 18.19.

A company's stock is selling for $4. The company has no outstanding debt. Analysts consider the liquidation value of the company to be at least $300,000 and there are 100,000 shares outstanding. What volatility smile would you expect to see?

In liquidation the company's stock price must be at least 300,000/100,000 = $3. The company's stock price should therefore always be at least $3. This means that the stock price distribution that has a thinner left tail and fatter right tail than the lognormal distribution. An upward sloping volatility smile can be expected.

Problem 18.20.

A company is currently awaiting the outcome of a major lawsuit. This is expected to be known within one month. The stock price is currently $20. If the outcome is positive, the stock price is expected to be $24 at the end of one month. If the outcome is negative, it is expected to be $18 at this time. The one-month risk-free interest rate is 8% per annum.

a. *What is the risk-neutral probability of a positive outcome?*
b. *What are the values of one-month call options with strike prices of $19, $20, $21, $22, and $23?*
c. *Use DerivaGem to calculate a volatility smile for one-month call options.*
d. *Verify that the same volatility smile is obtained for one-month put options.*

(a) If p is the risk-neutral probability of a positive outcome (stock price rises to $24), we must have

$$24p + 18(1 - p) = 20e^{0.08 \times 0.0833}$$

so that $p = 0.356$

(b) The price of a call option with strike price K is $(24 - K)pe^{-0.08 \times 0.08333}$ when $K < 24$. Call options with strike prices of 19, 20, 21, 22, and 23 therefore have prices 1.766, 1.413, 1.060, 0.707, and 0.353, respectively.

(c) From DerivaGem the implied volatilities of the options with strike prices of 19, 20, 21, 22, and 23 are 49.8%, 58.7%, 61.7%, 60.2%, and 53.4%, respectively. The volatility smile is therefore a "frown" with the volatilities for deep-out-of-the-money and deep-in-the-money options being lower than those for close-to-the-money options.

(d) The price of a put option with strike price K is $(K - 18)(1 - p)e^{-0.08 \times 0.08333}$. Put options with strike prices of 19, 20, 21, 22, and 23 therefore have prices of 0.640, 1.280, 1.920, 2.560, and 3.200. DerivaGem gives the implied volatilities as 49.81%, 58.68%, 61.69%, 60.21%, and 53.38%. Allowing for rounding errors these are the same as the implied volatilities for put options.

Problem 18.21.

A futures price is currently $40. The risk-free interest rate is 5%. Some news is expected tomorrow that will cause the volatility over the next three months to be either 10% or 30%. There is a 60% chance of the first outcome and a 40% chance of the second outcome. Use DerivaGem to calculate a volatility smile for three-month options.

The calculations are shown in the following table. For example, when the strike price is 34, the price of a call option with a volatility of 10% is 5.926, and the price of a call option when the volatility is 30% is 6.312. When there is a 60% chance of the first volatility and 40% of the second, the price is $0.6 \times 5.926 + 0.4 \times 6.312 = 6.080$. The implied volatility given by this price is 23.21. The table shows that the uncertainty about volatility leads to a classic volatility smile similar to that in Figure 18.1 of the text. In general when volatility is stochastic with the stock price and volatility uncorrelated we get a pattern of implied volatilities similar to that observed for currency options.

Strike Price	Call Option Price 10% Volatility	Call Option Price 30% Volatility	Weighted Price	Implied Volatility (%)
34	5.926	6.312	6.080	23.21
36	3.962	4.749	4.277	21.03
38	2.128	3.423	2.646	18.88
40	0.788	2.362	1.418	18.00
42	0.177	1.560	0.730	18.80
44	0.023	0.988	0.409	20.61
46	0.002	0.601	0.242	22.43

Problem 18.22.

Data for a number of foreign currencies are provided on the author's Web site:

http://www.rotman.utoronto.ca/~hull/data

Choose a currency and use the data to produce a table similar to Table 18.1.

The following table shows the percentage of daily returns greater than 1, 2, 3, 4, 5, and 6 standard deviations for each currency. The pattern is similar to that in Table 18.1.

	> 1sd	> 2sd	> 3sd	> 4sd	> 5sd	> 6sd
AUD	24.8	5.3	1.3	0.2	0.1	0.0
BEF	24.3	5.7	1.3	0.6	0.2	0.0
CHF	26.1	4.2	1.3	0.6	0.1	0.0
DEM	23.9	5.0	1.4	0.6	0.1	0.0
DKK	26.7	5.8	1.3	0.3	0.0	0.0
ESP	28.2	5.1	0.9	0.3	0.1	0.0
FRF	26.0	5.4	1.4	0.2	0.0	0.0
GBP	23.9	6.4	1.1	0.4	0.1	0.0
ITL	25.4	6.6	1.1	0.2	0.0	0.0
NLG	25.6	5.7	1.7	0.2	0.0	0.0
SEK	28.2	5.2	1.0	0.0	0.0	0.0
Normal	31.7	4.6	0.3	0.0	0.0	0.0

Problem 18.23.

Data for a number of stock indices are provided on the author's Web site:
http://www.rotman.utoronto.ca/~hull/data
Choose an index and test whether a three standard deviation down movement happens more often than a three standard deviation up movement.

The results are shown in the table below.

	> 3sd down	> 3sd up
TSE	0.88	0.22
S&P	0.55	0.44
FTSE	0.55	0.66
CAC	0.33	0.33
Nikkei	0.55	0.66
Total	0.57	0.46

Problem 18.24.

Consider a European call and a European put with the same strike price and time to maturity. Show that they change in value by the same amount when the volatility increases from a level, σ_1, to a new level, σ_2 within a short period of time. (Hint Use put–call parity.)

Define c_1 and p_1 as the values of the call and the put when the volatility is σ_1. Define c_2 and p_2 as the values of the call and the put when the volatility is σ_2. From put–call parity

$$p_1 + S_0 e^{-qT} = c_1 + Ke^{-rT}$$
$$p_2 + S_0 e^{-qT} = c_2 + Ke^{-rT}$$

If follows that

$$p_1 - p_2 = c_1 - c_2$$

Problem 18.25.

An exchange rate is currently 1.0 and the implied volatilities of six-month European options with strike prices 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, and 1.3 are 13%, 12%, 11%, 10%, 11%, 12%, and 13%. The domestic and foreign risk free rates are both 2.5%. Calculate the implied probability distribution using an approach similar to that used in the appendix for Example 18.2. Compare it with the implied distribution where all the implied volatilities are 11.5%.

Define:

$$g(S_T) = g_1 \text{ for } 0.7 \le S_T < 0.8$$
$$g(S_T) = g_2 \text{ for } 0.8 \le S_T < 0.9$$
$$g(S_T) = g_3 \text{ for } 0.9 \le S_T < 1.0$$
$$g(S_T) = g_4 \text{ for } 1.0 \le S_T < 1.1$$
$$g(S_T) = g_5 \text{ for } 1.1 \le S_T < 1.2$$
$$g(S_T) = g_6 \text{ for } 1.2 \le S_T < 1.3$$

The value of g_1 can be calculated by interpolating to get the implied volatility for a six-month option with a strike price of 0.75 as 12.5%. This means that options with strike prices of 0.7, 0.75, and 0.8 have implied volatilities of 13%, 12.5% and 12%, respectively. From DerivaGem their prices are \$0.2963, \$0.2469, and \$0.1976, respectively. Using equation (18A.1) with $K = 0.75$ and $\delta = 0.05$ we get

$$g_1 = \frac{e^{0.025\times 0.5}(0.2963 + 0.1976 - 2 \times 0.2469)}{0.05^2} = 0.0315$$

Similar calculations show that $g_2 = 0.7241$, $g_3 = 4.0788$, $g_4 = 3.6766$, $g_5 = 0.0.7285$, and $g_6 = 0.0898$. The total probability between 0.7 and 1.3 is the sum of these numbers multiplied by 0.1 or 0.9329. If the volatility had been flat at 11.5% the values of g_1, g_2, g_3, g_4, g_5, and g_6 would have been 0.0239, 0.9328, 4.2248, 3.7590, 0.9613, and 0.0938. The total probability between 0.7 and 1.3 is in this case 0.9996. This shows that the volatility smile gives rise to heavy tails for the distribution.

Problem 18.26.

Use Table 18.2 to calculate the implied volatility a trader would use for an 11-month option with $K/S_0 = 0.98$

Interpolation gives the volatility for a six-month option with a strike price of 98 as 12.82%. Interpolation also gives the volatility for a 12-month option with a strike price of 98 as 13.7%. A final interpolation gives the volatility of an 11-month option with a strike price of 98 as 13.55%. The same answer is obtained if the sequence in which the interpolations is done is reversed.

CHAPTER 19
Basic Numerical Procedures

Notes for the Instructor

Chapter 19 presents the standard numerical procedures used to value derivatives when analytic results are not available. These involve binomial/trinomial trees, Monte Carlo simulation, and finite difference methods.

Binomial trees are introduced in Chapter 11, and Section 19.1 and 19.2 can be regarded as a review and more in-depth treatment of that material. When covering Section 19.1, I usually go through in some detail the calculations for a number of nodes in an example such as the one in Figure 19.3. Once the basic tree building and roll back procedure has been covered it is fairly easy to explain how it can be extended to currencies, indices, futures, and stocks that pay dividends. Also the calculation of hedge statistics such as delta, gamma, and vega can be explained. The software DerivaGem is a convenient way of displaying trees in class as well as an important calculation tool for students.

The binomial tree and Monte Carlo simulation approaches use risk-neutral valuation arguments. By contrast, the finite difference method solves the underlying differential equation directly. However, as explained in the book the explicit finite difference method is essentially the same as the trinomial tree method and the implicit finite difference method is essentially the same as a multinomial tree approach where the are $M + 1$ branches emanating from each node. Binomial trees and finite difference methods are most appropriate for American options; Monte Carlo simulation is most appropriate for path-dependent options.

Any of Problems 19.25 to 19.30 work well as assignment questions.

QUESTIONS AND PROBLEMS

Problem 19.1.

Which of the following can be estimated for an American option by constructing a single binomial tree: delta, gamma, vega, theta, rho?

Delta, gamma, and theta can be determined from a single binomial tree. Vega is determined by making a small change to the volatility and recomputing the option price using a new tree. Rho is calculated by making a small change to the interest rate and recomputing the option price using a new tree.

Problem 19.2.

Calculate the price of a three-month American put option on a non-dividend-paying stock when the stock price is \$60, the strike price is \$60, the risk-free interest rate is 10%

per annum, and the volatility is 45% per annum. Use a binomial tree with a time interval of one month.

In this case, $S_0 = 60$, $K = 60$, $r = 0.1$, $\sigma = 0.45$, $T = 0.25$, and $\Delta t = 0.0833$. Also

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.45\sqrt{0.0833}} = 1.1387$$

$$d = \frac{1}{u} = 0.8782$$

$$a = e^{r\Delta t} = e^{0.1\times 0.0833} = 1.0084$$

$$p = \frac{a-d}{u-d} = 0.4998$$

$$1 - p = 0.5002$$

The output from DerivaGem for this example is shown in the Figure S19.1. The calculated price of the option is \$5.16.

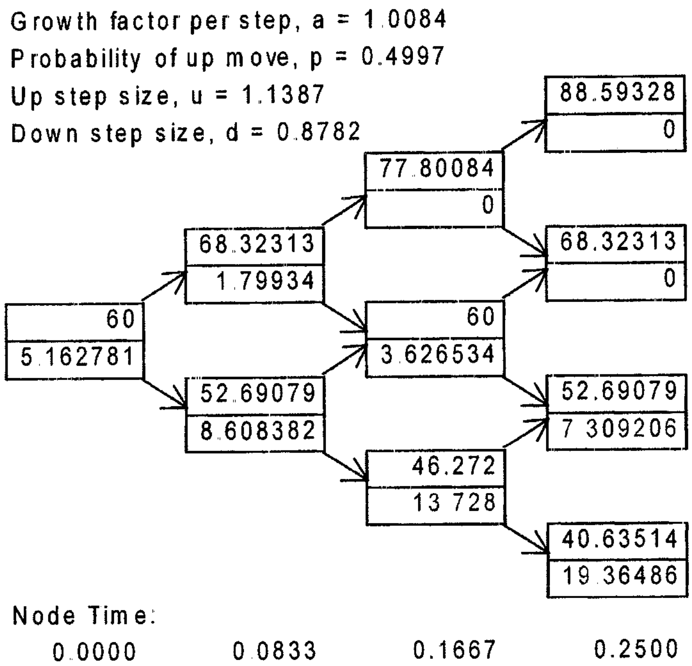


Figure S19.1 Tree for Problem 19.2

Problem 19.3.

Explain how the control variate technique is implemented when a tree is used to value American options.

The control variate technique is implemented by

(a) valuing an American option using a binomial tree in the usual way ($= f_A$).

(b) valuing the European option with the same parameters as the American option using the same tree ($= f_E$).

(c) valuing the European option using Black–Scholes ($= f_{BS}$). The price of the American option is estimated as $f_A + f_{BS} - f_E$.

Problem 19.4.

Calculate the price of a nine-month American call option on corn futures when the current futures price is 198 cents, the strike price is 200 cents, the risk-free interest rate is 8% per annum, and the volatility is 30% per annum. Use a binomial tree with a time interval of three months.

In this case $F_0 = 198$, $K = 200$, $r = 0.08$, $\sigma = 0.3$, $T = 0.75$, and $\Delta t = 0.25$. Also

$$
\begin{aligned}
u &= e^{0.3\sqrt{0.25}} = 1.1618 \\
d &= \frac{1}{u} = 0.8607 \\
a &= 1 \\
p &= \frac{a-d}{u-d} = 0.4626 \\
1 - p &= 0.5373
\end{aligned}
$$

The output from DerivaGem for this example is shown in the Figure S19.2. The calculated price of the option is 20.34 cents.

Problem 19.5.

Consider an option that pays off the amount by which the final stock price exceeds the average stock price achieved during the life of the option. Can this be valued using the binomial tree approach? Explain your answer.

A binomial tree cannot be used in the way described in this chapter. This is an example of what is known as a history-dependent option. The payoff depends on the path followed by the stock price as well as its final value. The option cannot be valued by starting at the end of the tree and working backward since the payoff at the final branches is not known unambiguously. Chapter 26 describes an extension of the binomial tree approach that can be used to handle options where the payoff depends on the average value of the stock price.

Problem 19.6.

"For a dividend-paying stock, the tree for the stock price does not recombine; but the tree for the stock price less the present value of future dividends does recombine." Explain this statement.

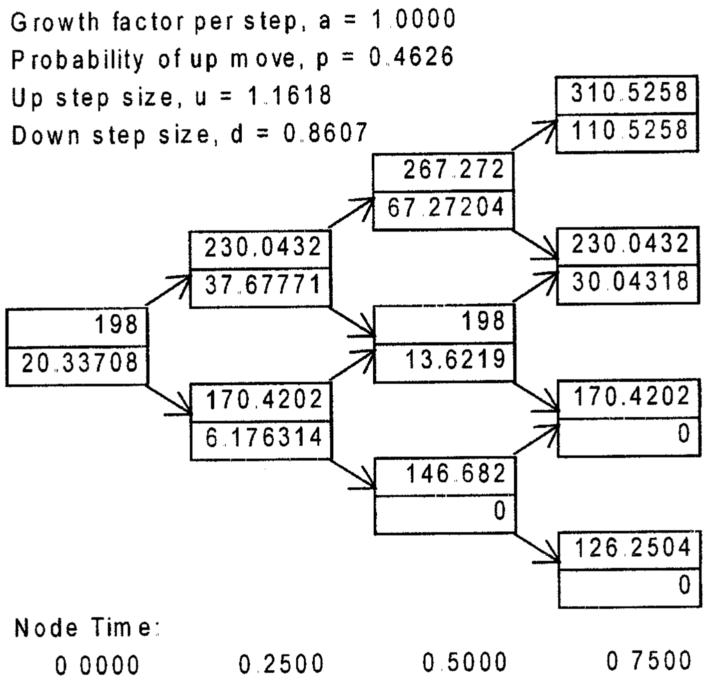


Figure S19.2 Tree for Problem 19.4

Suppose a dividend equal to D is paid during a certain time interval. If S is the stock price at the beginning of the time interval, it will be either $Su - D$ or $Sd - D$ at the end of the time interval. At the end of the next time interval, it will be one of $(Su - D)u$, $(Su - D)d$, $(Sd - D)u$ and $(Sd - D)d$. Since $(Su - D)d$ does not equal $(Sd - D)u$ the tree does not recombine. If S is equal to the stock price less the present value of future dividends, this problem is avoided.

Problem 19.7.

Show that the probabilities in a Cox, Ross, and Rubinstein binomial tree are negative when the condition in footnote 9 holds.

With the usual notation

$$p = \frac{a - d}{u - d}$$

$$1 - p = \frac{u - a}{u - d}$$

If $a < d$ or $a > u$, one of the two probabilities is negative. This happens when

$$e^{(r-q)\Delta t} < e^{-\sigma\sqrt{\Delta t}}$$

or

$$e^{(r-q)\Delta t} > e^{\sigma\sqrt{\Delta t}}$$

This in turn happens when $(q-r)\sqrt{\Delta t} > \sigma$ or $(r-q)\sqrt{\Delta t} > \sigma$ Hence negative probabilities occur when

$$\sigma < |(r-q)\sqrt{\Delta t}|$$

This is the condition in footnote 9.

Problem 19.8.

Use stratified sampling with 100 trials to improve the estimate of π in Business Snapshot 19.1 and Table 19.1.

In Table 19.1 cells A1, A2, A3,..., A100 are random numbers between 0 and 1 defining how far to the right in the square the dart lands. Cells B1, B2, B3,...,B100 are random numbers between 0 and 1 defining how high up in the square the dart lands. For stratified sampling we could choose equally spaced values for the A's and the B's and consider every possible combination. To generate 100 samples we need ten equally spaced values for the A's and the B's so that there are $10 \times 10 = 100$ combinations. The equally spaced values should be 0.05, 0.15, 0.25,..., 0.95. We could therefore set the A's and B's as follows:

$$A1 = A2 = A3 = ... = A10 = 0.05$$

$$A11 = A12 = A13 = ... = A20 = 0.15$$

$$\cdots$$

$$\cdots$$

$$A91 = A92 = A93 = ... = A100 = 0.95$$

and

$$B1 = B11 = B21 = ... = B91 = 0.05$$

$$B2 = B12 = B22 = ... = B92 = 0.15$$

$$\cdots$$

$$\cdots$$

$$B10 = B20 = B30 = ... = B100 = 0.95$$

We get a value for π equal to 3.2, which is closer to the true value than the value of 3.04 obtained with random sampling in Table 19.1. Because samples are not random we cannot easily calculate a standard error of the estimate.

Problem 19.9.

Explain why the Monte Carlo simulation approach cannot easily be used for American-style derivatives.

In Monte Carlo simulation sample values for the derivative security in a risk-neutral world are obtained by simulating paths for the underlying variables. On each simulation run, values for the underlying variables are first determined at time Δt, then at time $2\Delta t$, then at time $3\Delta t$, etc. At time $i\Delta t$ ($i = 0, 1, 2\ldots$) it is not possible to determine whether early exercise is optimal since the range of paths which might occur after time $i\Delta t$ have not been investigated. In short, Monte Carlo simulation works by moving forward from time t to time T. Other numerical procedures which accommodate early exercise work by moving backwards from time T to time t.

Problem 19.10.

A nine-month American put option on a non-dividend-paying stock has a strike price of \$49. The stock price is \$50, the risk-free rate is 5% per annum, and the volatility is 30% per annum. Use a three-step binomial tree to calculate the option price.

In this case, $S_0 = 50$, $K = 49$, $r = 0.05$, $\sigma = 0.30$, $T = 0.75$, and $\Delta t = 0.25$. Also

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}} = e^{0.30\sqrt{0.25}} = 1.0126 \\ d &= \frac{1}{u} = 0.8607 \\ a &= e^{r\Delta t} = e^{0.1\times 0.0833} = 1.0084 \\ p &= \frac{a-d}{u-d} = 0.5043 \\ 1-p &= 0.4957 \end{aligned}$$

The output from DerivaGem for this example is shown in the Figure S19.3. The calculated price of the option is \$4.29. Using 100 steps the price obtained is \$3.91

Problem 19.11.

Use a three-time-step tree to value a nine-month American call option on wheat futures. The current futures price is 400 cents, the strike price is 420 cents, the risk-free rate is 6%, and the volatility is 35% per annum. Estimate the delta of the option from your tree.

In this case $F_0 = 400$, $K = 420$, $r = 0.06$, $\sigma = 0.35$, $T = 0.75$, and $\Delta t = 0.25$. Also

$$\begin{aligned} u &= e^{0.35\sqrt{0.25}} = 1.1912 \\ d &= \frac{1}{u} = 0.8395 \\ a &= 1 \\ p &= \frac{a-d}{u-d} = 0.4564 \\ 1-p &= 0.5436 \end{aligned}$$

The output from DerivaGem for this example is shown in the Figure S19.4. The calculated price of the option is 42.07 cents. Using 100 time steps the price obtained is 38.64. The

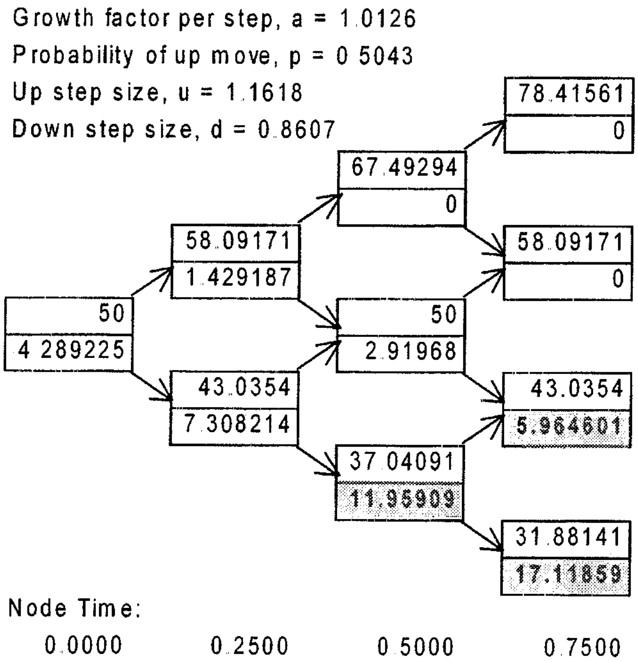


Figure S19.3 Tree for Problem 19.10

options delta is calculated from the tree is

$$(79.971 - 11.419)/(476.498 - 335.783) = 0.487$$

When 100 steps are used the estimate of the option's delta is 0.483.

Problem 19.12.

A three-month American call option on a stock has a strike price of \$20. The stock price is \$20, the risk-free rate is 3% per annum, and the volatility is 25% per annum. A dividend of \$2 is expected in 1.5 months. Use a three-step binomial tree to calculate the option price.

In this case the present value of the dividend is $2e^{-0.03\times0.125} = 1.9925$. We first build a tree for $S_0 = 20 - 1.9925 = 18.0075$, $K = 20$, $r = 0.03$, $\sigma = 0.25$, and $T = 0.25$ with $\Delta t = 0.08333$. This gives Figure S19.5. For nodes between times 0 and 1.5 months we then add the present value of the dividend to the stock price. The result is the tree in

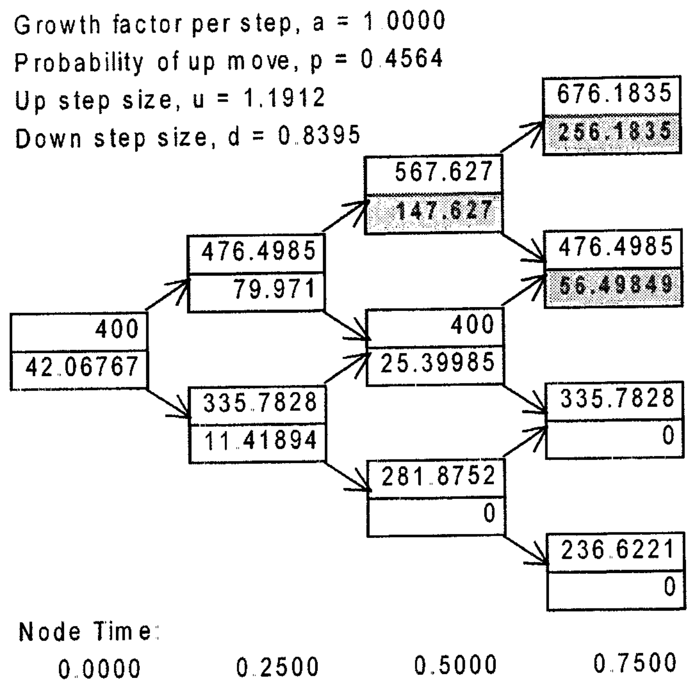


Figure S19.4 Tree for Problem 19.11

Figure S19.6. The price of the option calculated from the tree is 0.674. When 100 steps are used the price obtained is 0.690.

Problem 19.13.

A one-year American put option on a non-dividend-paying stock has an exercise price of \$18. The current stock price is \$20, the risk-free interest rate is 15% per annum, and the volatility of the stock price is 40% per annum. Use the DerivaGem software with four 3-month time steps to estimate the value of the option. Display the tree and verify that the option prices at the final and penultimate nodes are correct. Use DerivaGem to value the European version of the option. Use the control variate technique to improve your estimate of the price of the American option.

In this case $S_0 = 20$, $K = 18$, $r = 0.15$, $\sigma = 0.40$, $T = 1$, and $\Delta t = 0.25$. The

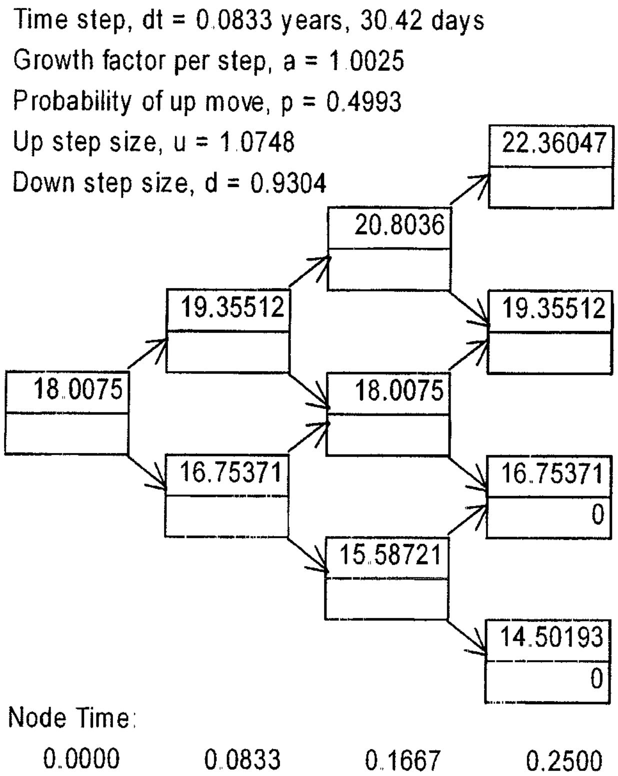


Figure S19.5 First tree for Problem 19.12

parameters for the tree are

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.4\sqrt{0.25}} = 1.2214$$
$$d = 1/u = 0.8187$$
$$a = e^{r\Delta t} = 1.0382$$
$$p = \frac{a-d}{u-d} = \frac{1.0382 - 0.8187}{1.2214 - 0.8187} = 0.545$$

The tree produced by DerivaGem for the American option is shown in Figure S19.7. The estimated value of the American option is $1.29.

As shown in Figure S19.8, the same tree can be used to value a European put option with the same parameters. The estimated value of the European option is $1.14. The option parameters are $S = 20$, $K = 18$, $r = 0.15$, $\sigma = 0.40$ and $T = 1$

$$d_1 = \frac{\ln(20/18) + 0.15 + 0.40^2/2}{0.40} = 0.8384$$
$$d_2 = d_1 - 0.40 = 0.4384$$

$$N(-d_1) = 0.2009; \quad N(-d_2) = 0.3306$$

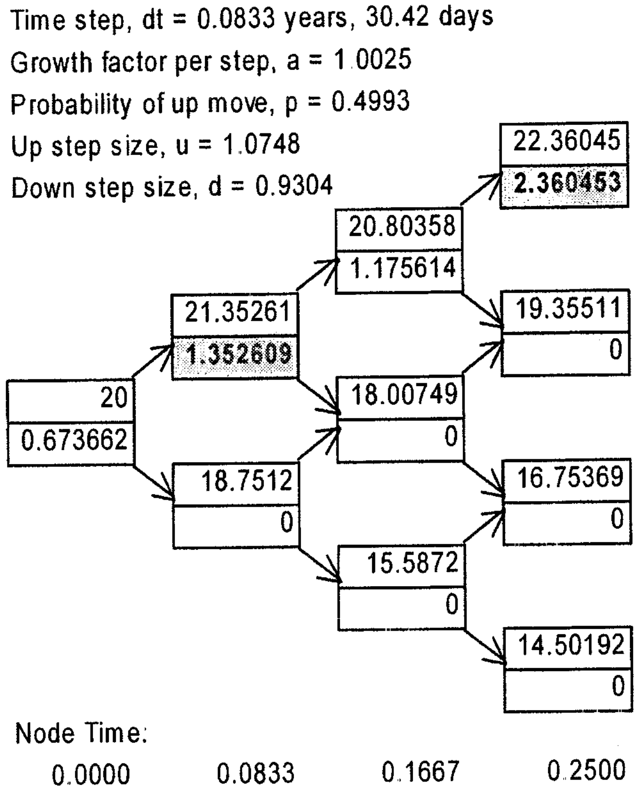


Figure S19.6 Final Tree for Problem 19.12

The true European put price is therefore

$$18e^{-0.15} \times 0.3306 - 20 \times 0.2009 = 1.10$$

The control variate estimate of the American put price is therefore $1.29 + 1.10 - 1.14 =$ \$1.25.

Problem 19.14.

A two-month American put option on a stock index has an exercise price of 480. The current level of the index is 484, the risk-free interest rate is 10% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 25% per annum. Divide the life of the option into four half-month periods and use the tree approach to estimate the value of the option.

In this case $S_0 = 484$, $K = 480$, $r = 0.10$, $\sigma = 0.25$ $q = 0.03$, $T = 0.1667$, and

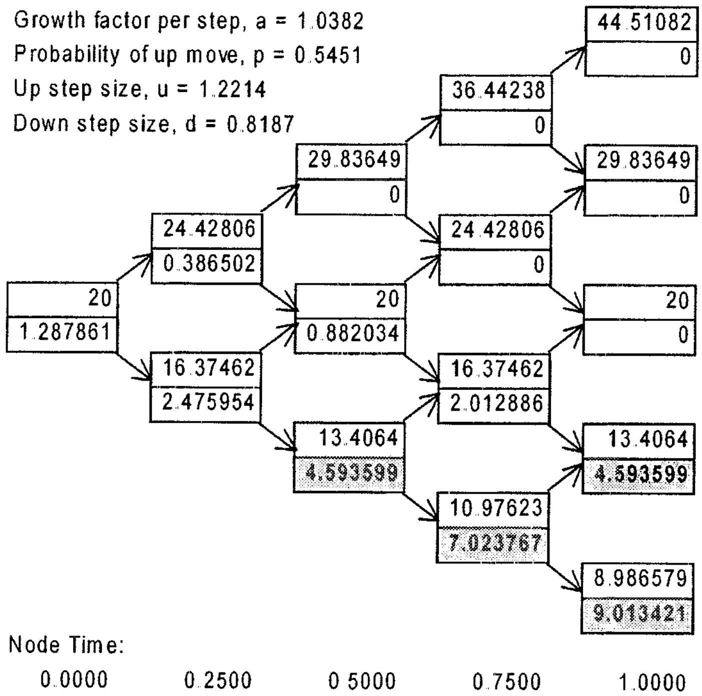


Figure S19.7 Tree to evaluate American option for Problem 19.13

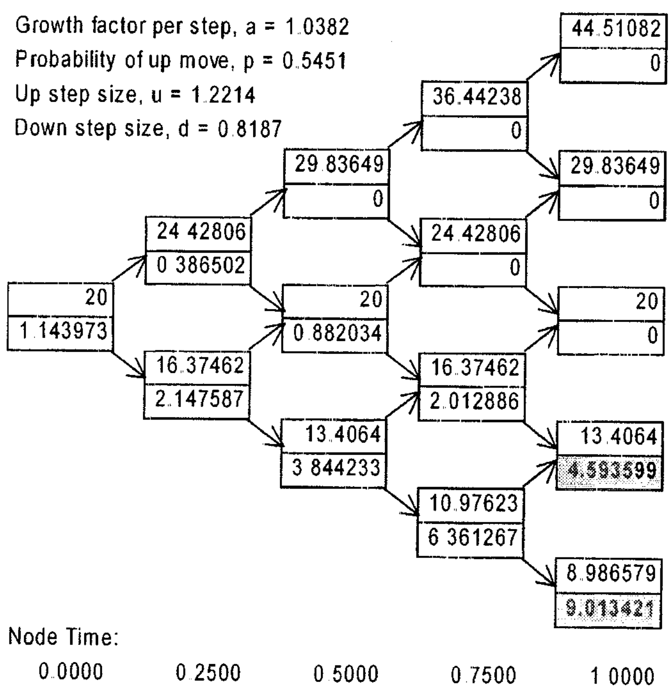


Figure S19.8 Tree to evaluate European option in Problem 19.13

$\Delta t = 0.04167$

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.25\sqrt{0.04167}} = 1.0524$$

$$d = \frac{1}{u} = 0.9502$$

$$a = e^{(r-q)\Delta t} = 1.00292$$

$$p = \frac{a-d}{u-d} = \frac{1.0029 - 0.9502}{1.0524 - 0.9502} = 0.516$$

The tree produced by DerivaGem is shown in the Figure S19.9. The estimated price of the option is $14.93.

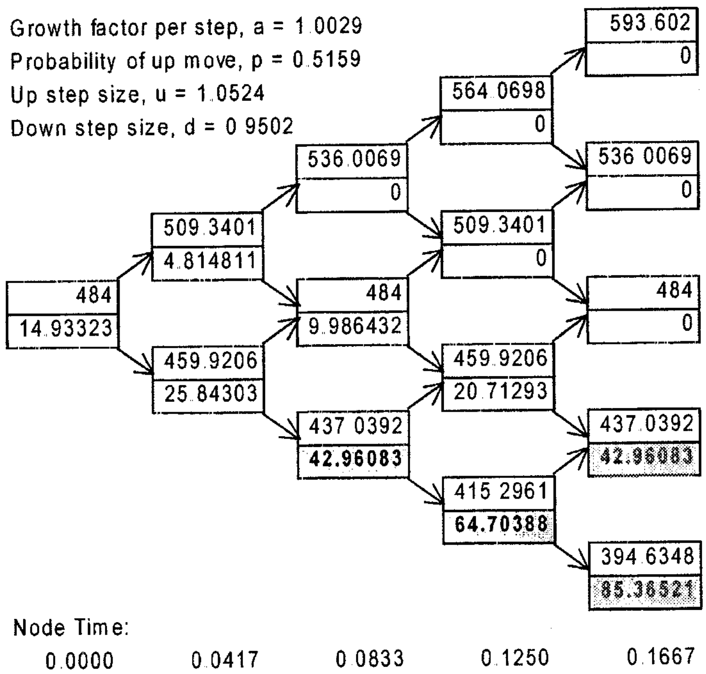


Figure S19.9 Tree to evaluate option in Problem 19.14

Problem 19.15.

How can the control variate approach improve the estimate of the delta of an American option when the tree approach is used?

First the delta of the American option is estimated in the usual way from the tree. Denote this by Δ_A^*. Then the delta of a European option which has the same parameters as the American option is calculated in the same way using the same tree. Denote this by Δ_B^*. Finally the true European delta, Δ_B, is calculated using the formulas in Chapter 17. The control variate estimate of delta is then:

$$\Delta_A^* - \Delta_B^* + \Delta_B$$

Problem 19.16.

Suppose that Monte Carlo simulation is being used to evaluate a European call option on a non-dividend-paying stock when the volatility is stochastic. How could the control variate and antithetic variable technique be used to improve numerical efficiency? Explain why it is necessary to calculate six values of the option in each simulation trial when both the control variate and the antithetic variable technique are used.

In this case a simulation requires two sets of samples from standardized normal distributions. The first is to generate the volatility movements. The second is to generate the stock price movements once the volatility movements are known. The control variate technique involves carrying out a second simulation on the assumption that the volatility is constant. The same random number stream is used to generate stock price movements as in the first simulation. An improved estimate of the option price is

$$f_A^* - f_B^* + f_B$$

where f_A^* is the option value from the first simulation (when the volatility is stochastic), f_B^* is the option value from the second simulation (when the volatility is constant) and f_B is the true Black-Scholes value when the volatility is constant.

To use the antithetic variable technique, two sets of samples from standardized normal distributions must be used for each of volatility and stock price. Denote the volatility samples by $\{V_1\}$ and $\{V_2\}$ and the stock price samples by $\{S_1\}$ and $\{S_2\}$. $\{V_1\}$ is antithetic to $\{V_2\}$ and $\{S_1\}$ is antithetic to $\{S_2\}$. Thus if

$$\{V_1\} = +0.83, +0.41, -0.21\ldots$$

then

$$\{V_2\} = -0.83, -0.41, +0.21\ldots$$

Similarly for $\{S_1\}$ and $\{S_2\}$.

An efficient way of proceeding is to carry out six simulations in parallel:

Simulation 1: Use $\{S_1\}$ with volatility constant
Simulation 2: Use $\{S_2\}$ with volatility constant
Simulation 3: Use $\{S_1\}$ and $\{V_1\}$
Simulation 4: Use $\{S_1\}$ and $\{V_2\}$
Simulation 5: Use $\{S_2\}$ and $\{V_1\}$
Simulation 6: Use $\{S_2\}$ and $\{V_2\}$

If f_i is the option price from simulation i, simulations 3 and 4 provide an estimate $0.5(f_3 + f_4)$ for the option price. When the control variate technique is used we combine this estimate with the result of simulation 1 to obtain $0.5(f_3 + f_4) - f_1 + f_B$ as an estimate of the price where f_B is, as above, the Black-Scholes option price. Similarly simulations 2, 5 and 6 provide an estimate $0.5(f_5 + f_6) - f_2 + f_B$. Overall the best estimate is:

$$0.5[0.5(f_3 + f_4) - f_1 + f_B + 0.5(f_5 + f_6) - f_2 + f_B]$$

Problem 19.17.

Explain how equations (19.27) to (19.30) change when the implicit finite difference method is being used to evaluate an American call option on a currency.

For an American call option on a currency

$$\frac{\partial f}{\partial t} + (r - r_f)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

With the notation in the text this becomes

$$\frac{f_{i+1,j} - f_{ij}}{\Delta t} + (r - r_f)j\Delta S \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} + \frac{1}{2}\sigma^2 j^2 \Delta S^2 \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\Delta S^2} = rf_{ij}$$

for $j = 1, 2 \ldots M - 1$ and $i = 0, 1 \ldots N - 1$. Rearranging terms we obtain

$$a_j f_{i,j-1} + b_j f_{ij} + c_j f_{i,j+1} = f_{i+1,j}$$

where

$$\begin{aligned} a_j &= \frac{1}{2}(r - r_f)j\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t \\ b_j &= 1 + \sigma^2 j^2 \Delta t + r\Delta t \\ c_j &= -\frac{1}{2}(r - r_f)j\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t \end{aligned}$$

Equations (19.28), (19.29) and (19.30) become

$$\begin{aligned} f_{Nj} &= \max\,[j\Delta S - K,\, 0] \quad j = 0, 1 \ldots M \\ f_{i0} &= 0 \quad i = 0, 1 \ldots N \\ f_{iM} &= M\Delta S - K \quad i = 0, 1 \ldots N \end{aligned}$$

Problem 19.18.

An American put option on a non-dividend-paying stock has four months to maturity. The exercise price is \$21, the stock price is \$20, the risk-free rate of interest is 10% per annum, and the volatility is 30% per annum. Use the explicit version of the finite difference approach to value the option. Use stock price intervals of \$4 and time intervals of one month.

We consider stock prices of \$0, \$4, \$8, \$12, \$16, \$20, \$24, \$28, \$32, \$36 and \$40. Using equation (19.34) with $r = 0.10$, $\Delta t = 0.0833$, $\Delta S = 4$, $\sigma = 0.30$, $K = 21$, $T = 0.3333$ we obtain the table shown below. The option price is \$1.56.

Grid for Finite Difference Approach in Problem 19.18.

Stock Price ($)	Time To Maturity (Months) 4	3	2	1	0
40	0.00	0.00	0.00	0.00	0.00
36	0.00	0.00	0.00	0.00	0.00
32	0.01	0.00	0.00	0.00	0.00
28	0.07	0.04	0.02	0.00	0.00
24	0.38	0.30	0.21	0.11	0.00
20	1.56	1.44	1.31	1.17	1.00
16	5.00	5.00	5.00	5.00	5.00
12	9.00	9.00	9.00	9.00	9.00
8	13.00	13.00	13.00	13.00	13.00
4	17.00	17.00	17.00	17.00	17.00
0	21.00	21.00	21.00	21.00	21.00

Problem 19.19.

The spot price of copper is $0.60 per pound. Suppose that the futures prices (dollars per pound) are as follows:

3 months	*0.59*
6 months	*0.57*
9 months	*0.54*
12 months	*0.50*

The volatility of the price of copper is 40% per annum and the risk-free rate is 6% per annum. Use a binomial tree to value an American call option on copper with an exercise price of $0.60 and a time to maturity of one year. Divide the life of the option into four 3-month periods for the purposes of constructing the tree.(Hint: As explained in Section 14.7, the futures price of a variable is its expected future price in a risk-neutral world.)

In this case $\Delta t = 0.25$ and $\sigma = 0.4$ so that

$$u = e^{0.4\sqrt{0.25}} = 1.2214$$

$$d = \frac{1}{u} = 0.8187$$

The futures prices provide estimates of the growth rate in copper in a risk-neutral world. During the first three months this growth rate (with continuous compounding) is

$$4 \ln \frac{0.59}{0.60} = -6.72\% \text{ per annum}$$

The parameter p for the first three months is therefore

$$\frac{e^{-0.0672\times0.25} - 0.8187}{1.2214 - 0.8187} = 0.4088$$

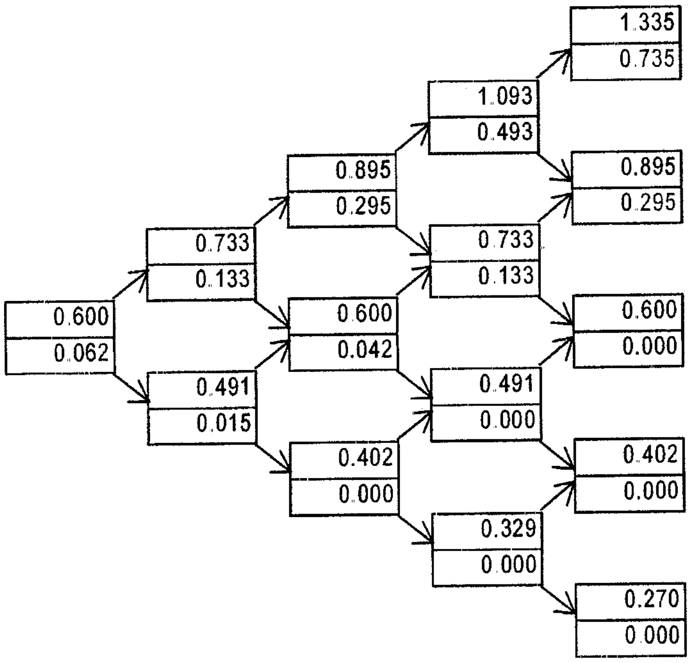


Figure S19.10 Tree to value option in Problem 19.19: At each node, upper number is price of copper and lower number is option price

The growth rate in copper is equal to −13.79%, −21.63% and −30.78% in the following three quarters. Therefore, the parameter p for the second three months is

$$\frac{e^{-0.1379\times 0.25} - 0.8187}{1.2214 - 0.8187} = 0.3660$$

For the third quarter it is

$$\frac{e^{-0.2163\times 0.25} - 0.8187}{1.2214 - 0.8187} = 0.3195$$

For the final quarter, it is

$$\frac{e^{-0.3078\times 0.25} - 0.8187}{1.2214 - 0.8187} = 0.2663$$

The tree for the movements in copper prices in a risk-neutral world is shown in Figure S19.10. The value of the option is \$0.062.

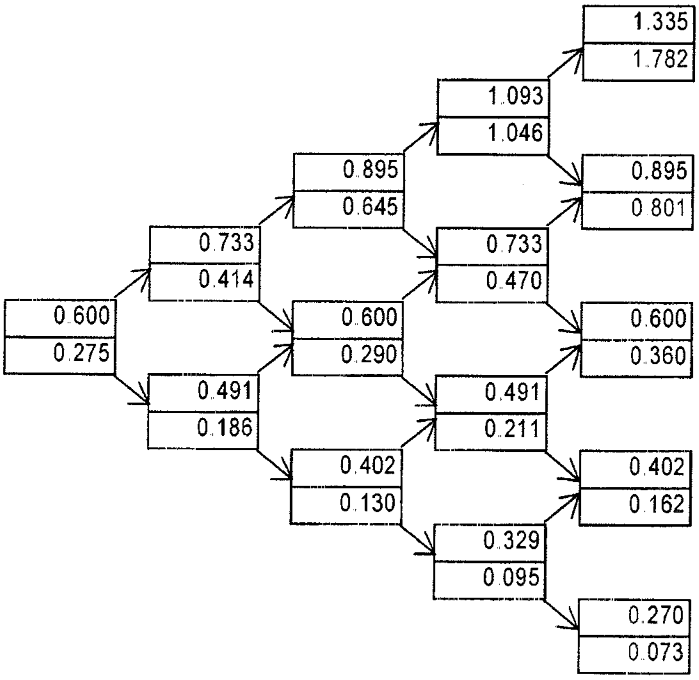


Figure S19.11 Tree to value derivative in Problem 19.20. At each node, upper number is price of copper and lower number is derivative security price.

Problem 19.20.

Use the binomial tree in Problem 19.19 to value a security that pays off x^2 in one year where x is the price of copper.

In this problem we use exactly the same tree for copper prices as in Problem 19.19. However, the values of the derivative are different. On the final nodes the values of the derivative equal the square of the price of copper. On other nodes they are calculated in the usual way. The current value of the security is \$0.275 (see Figure S19.11).

Problem 19.21.

When do the boundary conditions for $S = 0$ and $S \longrightarrow \infty$ affect the estimates of derivative prices in the explicit finite difference method?

Define S_t as the current asset price, $S_{\max}$ as the highest asset price considered and $S_{\min}$ as the lowest asset price considered. (In the example in the text $S_{\min} = 0$). Let

$$Q_1 = \frac{S_{\max} - S_t}{\Delta S} \quad \text{and} \quad Q_2 = \frac{S_t - S_{\min}}{\Delta S}$$

and let N be the number of time intervals considered. From the structure of the calculations in the explicit version of the finite difference method, we can see that the values assumed for the derivative security at $S = S_{\min}$ and $S = S_{\max}$ affect the derivative security's value at time t if

$$N \geq \max(Q_1, Q_2)$$

Problem 19.22.

How would you use the antithetic variable method to improve the estimate of the European option in Business Snapshot 19.2 and Table 19.2?

The following changes could be made. Set LI as

= NORMSINV(RAND())

A1 as

=C*EXP(($E$2-$F$2*$F$2/2)*$G$2+$F$2*L2*SQRT($G$2))

H1 as

=C*EXP(($E$2-$F$2*$F$2/2)*$G$2-$F$2*L2*SQRT($G$2)).

I1 as

= EXP(-$E$2*$G$2)*MAX(H1-$D$2,0)

and J1 as

=0.5*(B1+J1)

Other entries in columns L, A, H, and I are defined similarly. The estimate of the value of the option is the average of the values in the J column

Problem 19.23.

A company has issued a three-year convertible bond that has a face value of $25 and can be exchanged for two of the company's shares at any time. The company can call the issue when the share price is greater than or equal to $18. Assuming that the company will force conversion at the earliest opportunity, what are the boundary conditions for the price of the convertible? Describe how you would use finite difference methods to value the convertible assuming constant interest rates. Assume there is no risk of the company defaulting.

The basic approach is similar to that described in Section 19.8. The only difference is the boundary conditions. For a sufficiently small value of the stock price, $S_{\min}$, it can be assumed that conversion will never take place and the convertible can be valued as a straight bond. The highest stock price which needs to be considered, $S_{\max}$, is $18. When this is reached the value of the convertible bond is $36. At maturity the convertible is worth the greater of $2S_T$ and $25 where S_T is the stock price.

The convertible can be valued by working backwards through the grid using either the explicit or the implicit finite difference method in conjunction with the boundary conditions. In formulas (19.25) and (19.32) the present value of the income on the convertible between time $t + i\,\Delta t$ and $t + (i+1)\,\Delta t$ discounted to time $t + i\,\Delta t$ must be added to the right-hand side. Chapter 26 considers the pricing of convertibles in more detail.

Problem 19.24.

Provide formulas that can be used for obtaining three random samples from standard normal distributions when the correlation between sample i and sample j is $\rho_{i,j}$.

Suppose x_1, x_2, and x_3 are random samples from three independent normal distributions. Random samples with the required correlation structure are ϵ_1, ϵ_2, ϵ_3 where

$$\epsilon_1 = x_1$$

$$\epsilon_2 = \rho_{12}x_1 + x_2\sqrt{1-\rho_{12}^2}$$

and

$$\epsilon_3 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$$

where

$$\alpha_1 = \rho_{13}$$

$$\alpha_1\rho_{12} + \alpha_2\sqrt{1-\rho_{12}^2} = \rho_{23}$$

and

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$$

This means that

$$\alpha_1 = \rho_{13}$$

$$\alpha_2 = \frac{\rho_{23} - \rho_{13}\rho_{12}}{\sqrt{1-\rho_{12}^2}}$$

$$\alpha_3 = \sqrt{1-\alpha_1^2-\alpha_2^2}$$

ASSIGNMENT QUESTIONS

Problem 19.25.

An American put option to sell a Swiss franc for dollars has a strike price of $0.80 and a time to maturity of one year. The volatility of the Swiss franc is 10%, the dollar interest rate is 6%, the Swiss franc interest rate is 3%, and the current exchange rate is 0.81. Use a three-time-step tree to value the option. Estimate the delta of the option from your tree.

The binomial tree is shown in Figure M19.1. The value of the option is estimated as 0.0207. and its delta is estimated as

$$\frac{0.006221 - 0.041153}{0.858142 - 0.764559} = -0.3733$$

Problem 19.26.

A one-year American call option on silver futures has an exercise price of \$9.00. The current futures price is \$8.50, the risk-free rate of interest is 12% per annum, and the volatility of the futures price is 25% per annum. Use the DerivaGem software with four three-month time steps to estimate the value of the option. Display the tree and verify that the option prices at the final and penultimate nodes are correct. Use DerivaGem to value the European version of the option. Use the control variate technique to improve your estimate of the price of the American option.

In this case $F_0 = 8.5$, $K = 9$, $r = 0.12$, $T = 1$, $\sigma = 0.25$, and $\Delta t = 0.25$. The parameters for the tree are

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.25\sqrt{0.25}} = 1.1331$$

$$d = \frac{1}{u} = 0.8825$$

$$a = 1$$

$$p = \frac{a-d}{u-d} = \frac{1-0.8825}{1.1331-0.8825} = 0.469$$

The tree output by DerivaGem for the American option is shown in Figure M19.2. The estimated value of the option is \$0.596. The tree produced by DerivaGem for the European version of the option is shown in Figure M19.3. The estimated value of the option is \$0.586. The Black–Scholes price of the option is \$0.570. The control variate estimate of the price of the option is therefore

$$0.596 + 0.570 - 0.586 = 0.580$$

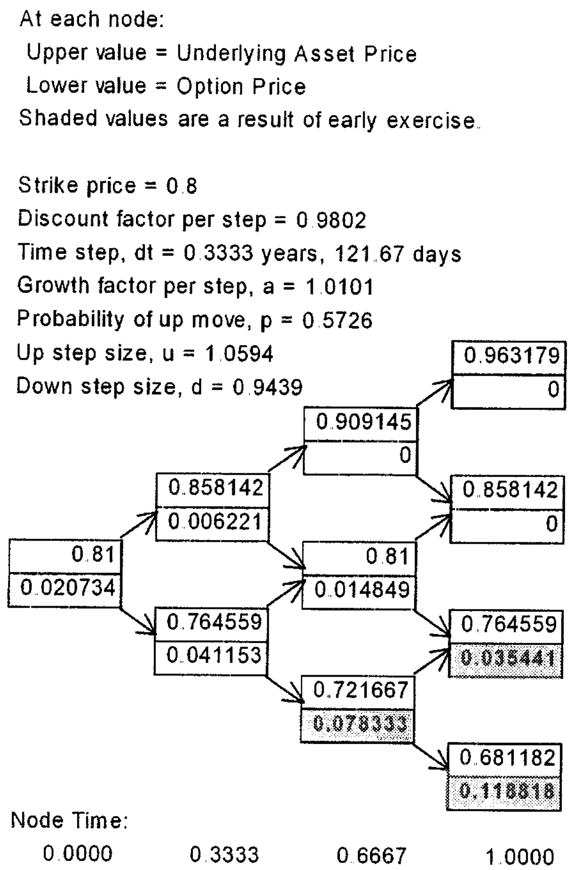


Figure M19.1 Tree for Problem 19.25

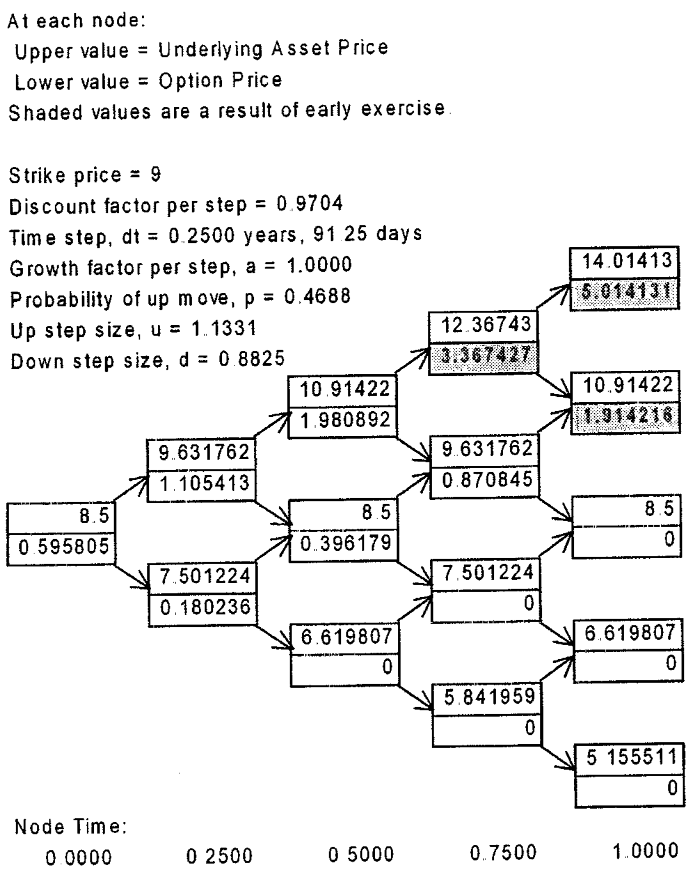


Figure M19.2 Tree for American option in Problem 19.26

Problem 19.27.

A six-month American call option on a stock is expected to pay dividends of \$1 per share at the end of the second month and the fifth month. The current stock price is \$30, the exercise price is \$34, the risk-free interest rate is 10% per annum, and the volatility of the part of the stock price that will not be used to pay the dividends is 30% per annum. Use the DerivaGem software with the life of the option divided into six time steps to estimate the value of the option. Compare your answer with that given by Black's approximation (see Section 13.12).

DerivaGem gives the value of the option as 0.989. Black's approximation sets the price of the American call option equal to the maximum of two European options. The first lasts the full six months. The second expires just before the final ex-dividend date. In this case the software shows that the first European option is worth 0.957 and the second is worth 0.997. Black's model therefore estimates the value of the American option as 0.997. This is close to the tree value of 0.989.

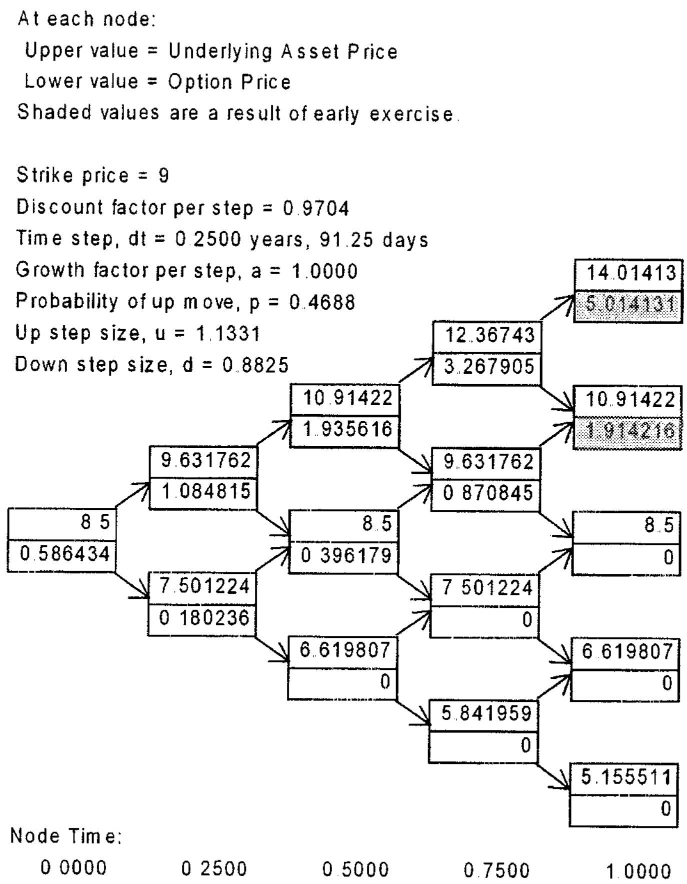


Figure M19.3 Tree for European option in Problem 19.26

Problem 19.28.

The current value of the British pound is \$1.60 and the volatility of the pound–dollar exchange rate is 15% per annum. An American call option has an exercise price of \$1.62 and a time to maturity of one year. The risk-free rates of interest in the United States and the United Kingdom are 6% per annum and 9% per annum, respectively. Use the explicit finite difference method to value the option. Consider exchange rates at intervals of 0.20 between 0.80 and 2.40 and time intervals of 3 months.

In this case equation (19.34) becomes

$$f_{ij} = a_j^* f_{i+1,j-1} + b_j^* f_{i+1,j} + c_j^* f_{i+1,j+1}$$

where

$$a_j^* = \frac{1}{1+r\Delta t}\left[-\frac{1}{2}(r-r_f)j\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t\right]$$
$$b_j^* = \frac{1}{1+r\Delta t}(1-\sigma^2 j^2 \Delta t)$$
$$c_j^* = \frac{1}{1+r\Delta t}\left[\frac{1}{2}(r-r_f)j\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t\right]$$

The parameters are $r = 0.06$, $r_f = 0.09$, $\sigma = 0.15$, $S = 1.60$, $K = 1.62$, $T = 1$, $\Delta t = 0.25$, $\Delta S = 0.2$ and we obtain the table shown below. The option price is $0.062.

Stock Price	Time To Maturity (Months)				
	12	9	6	3	0
2.40	0.780	0.780	0.780	0.780	0.780
2.20	0.580	0.580	0.580	0.580	0.580
2.00	0.380	0.380	0.380	0.380	0.380
1.80	0.180	0.180	0.180	0.180	0.180
1.60	0.062	0.054	0.043	0.027	0.000
1.40	0.011	0.007	0.003	0.000	0.000
1.20	0.001	0.000	0.000	0.000	0.000
1.00	0.000	0.000	0.000	0.000	0.000
0.80	0.000	0.000	0.000	0.000	0.000

Problem 19.29.

Answer the following questions concerned with the alternative procedures for constructing trees in Section 19.4.

a. *Show that the binomial model in Section 19.4 is exactly consistent with the mean and variance of the change in the logarithm of the stock price in time Δt.*

b. *Show that the trinomial model in Section 19.4 is consistent with the mean and variance of the change in the logarithm of the stock price in time Δt when terms of order $(\Delta t)^2$ and higher are ignored.*

c. *Construct an alternative to the trinomial model in Section 19.4 so that the probabilities are 1/6, 2/3, and 1/6 on the upper, middle, and lower branches emanating from each node. Assume that the branching is from S to Su, Sm, or Sd with $m^2 = ud$. Match the mean and variance of the change in the logarithm of the stock price exactly.*

(a) For the binomial model in Section 19.4 there are two equally likely changes in the logarithm of the stock price in a time step of length Δt. These are $(r-\sigma^2/2)\Delta t+\sigma\sqrt{\Delta t}$ and $(r-\sigma^2/2)\Delta t - \sigma\sqrt{\Delta t}$. The expected change in the logarithm of the stock price is

$$0.5[(r-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}] + 0.5[(r-\sigma^2/2)\Delta t - \sigma\sqrt{\Delta t}] = (r-\sigma^2/2)\Delta t$$

This is correct. The variance of the change in the logarithm of the stock price is

$$0.5\sigma^2\Delta t + 0.5\sigma^2\Delta t = \sigma^2\Delta t$$

This is correct.

(b) For the trinomial tree model in Section 19.4, the change in the logarithm of the stock price in a time step of length Δt is $+\sigma\sqrt{3\Delta t}$, 0, and $-\sigma\sqrt{3\Delta t}$ with probabilities

$$\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r-\frac{\sigma^2}{2}\right)+\frac{1}{6}, \qquad \frac{2}{3}, \qquad -\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r-\frac{\sigma^2}{2}\right)+\frac{1}{6}$$

The expected change is

$$\left(r-\frac{\sigma^2}{2}\right)\Delta t$$

It variance is $\sigma^2\Delta t$ plus a term of order $(\Delta t)^2$. These are correct.

(c) To get the expected change in the logarithm of the stock price in time Δt correct we require

$$\frac{1}{6}(\ln u)+\frac{2}{3}(\ln m)+\frac{1}{6}(\ln d)=\left(r-\frac{\sigma^2}{2}\right)\Delta t$$

The relationship $m^2 = ud$ implies $\ln m = 0.5(\ln u + \ln d)$ so that the requirement becomes

$$\ln m = \left(r-\frac{\sigma^2}{2}\right)\Delta t$$

or

$$m = e^{(r-\sigma^2)\Delta t}$$

The expected change in $\ln S$ is $\ln m$. To get the variance of the change in the logarithm of the stock price in time Δt correct we require

$$\frac{1}{6}(\ln u - \ln m)^2 + \frac{1}{6}(\ln d - \ln m)^2 = \sigma^2\Delta t$$

Because $\ln u - \ln m = -(\ln d - \ln m)$ it follows that

$$\ln u = \ln m + \sigma\sqrt{3\Delta t}$$

$$\ln d = \ln m - \sigma\sqrt{3\Delta t}$$

These results imply that

$$m = e^{(r-\sigma^2)\Delta t}$$

$$u = e^{(r-\sigma^2)\Delta t + \sigma\sqrt{3\Delta t}}$$

$$d = e^{(r-\sigma^2)\Delta t - \sigma\sqrt{3\Delta t}}$$

Problem 19.30.

The DerivaGem Application Buider functions enable you to investigate how the prices of options calculated from a binomial tree converge to the correct value as the number of time steps increases. (See Figure 19.4 and Sample Application A in DerivaGem.) Consider a put option on a stock index where the index level is 900, the strike price is 900, the risk-free rate is 5%, the dividend yield is 2%, and the time to maturity is 2 years

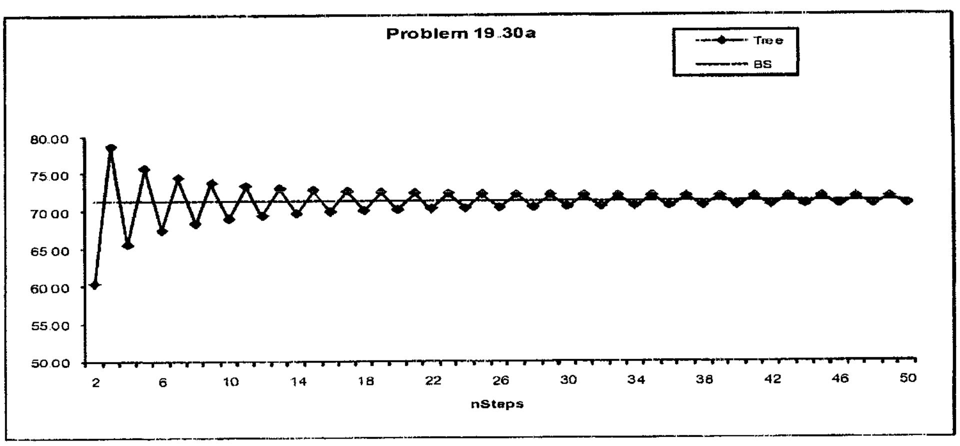
a. *Produce results similar to Sample Application A on convergence for the situation where the option is European and the volatility of the index is 20%.*

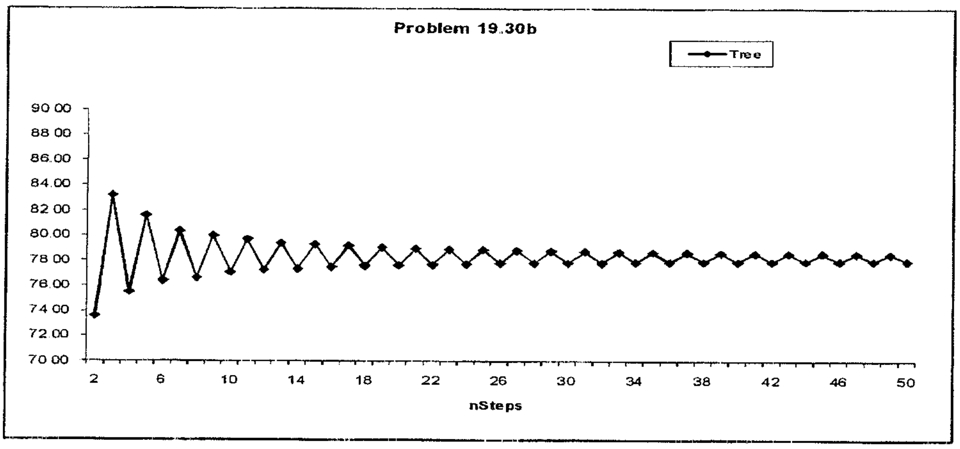
b. *Produce results similar to Sample Application A on convergence for the situation where the option is American and the volatility of the index is 20%.*

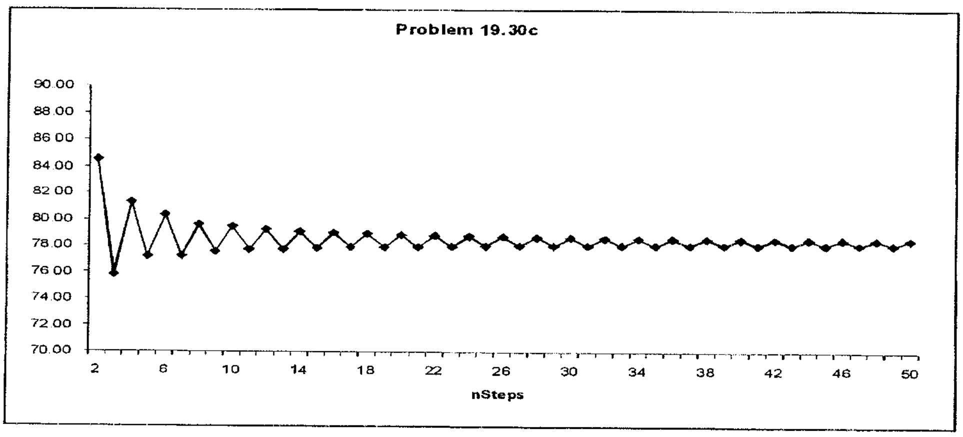
c. *Produce a chart showing the pricing of the American option when the volatility is 20% as a function of the number of time steps when the control variate technique is used.*

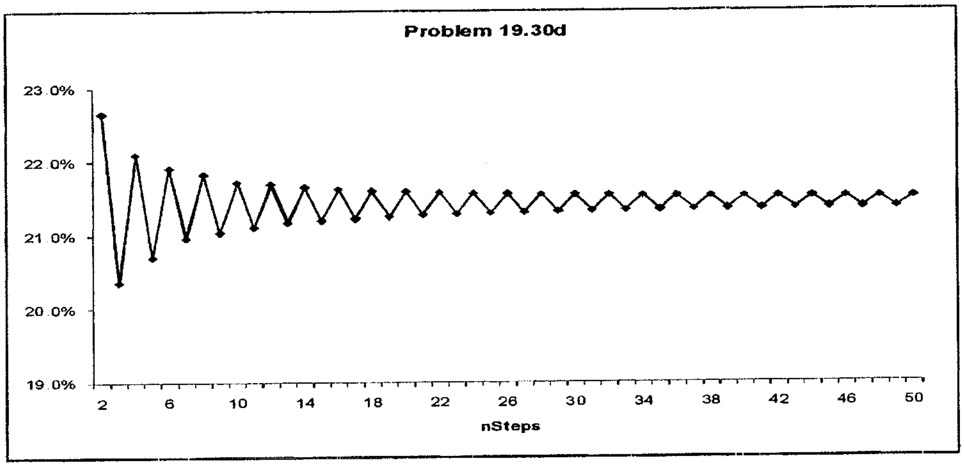
d. *Suppose that the price of the American option in the market is 85.0. Produce a chart showing the implied volatility estimate as a function of the number of time steps.*

See the charts following.




Problem 19.30b
Tree
90.00
88.00
86.00
84.00
82.00
80.00
78.00
76.00
74.00
72.00
70.00
2
6
10
14
18
22
26
30
34
38
42
46
50
nSteps


Problem 19.30c
90.00
88.00
86.00
84.00
82.00
80.00
78.00
76.00
74.00
72.00
70.00
2
6
10
14
18
22
26
30
34
38
42
46
50
nSteps



Problem 19.30d
23.0%
22.0%
21.0%
20.0%
19.0%
2
6
10
14
18
22
26
30
34
38
42
46
50
nSteps

CHAPTER 20
Value at Risk

Notes for the Instructor

Some instructors may prefer to cover Chapter 21 before Chapter 20 because the estimation of volatilities and correlations is necessary for the model building approach for calculating VaR. This works well. I prefer to do Chapter 20 first because VaR has become such a fundamental measure. After doing Chapter 20 students understand why Chapter 21 is important.

When the model building approach is covered it should be emphasized that we are relating the *actual change* in the value of the portfolio to *percentage changes* in the values of the market variables. The use of the model building approach can be presented in the context of the classic work of Markowitz on portfolio selection. I like to spend some time on the simple examples in Section 20.3. This leads on to the linear model in Section 20.4. Tables 20.3 and 20.4 form a starting point for discussing principal components analysis and provide the data for a number of end-of-chapter problems. Principal components analysis is also used in the discussion of interest rate models in Chapter 31.

Any of Problems 20.16 to 20.21 work well as assignment questions. Problem 20.20 is relatively challenging and requires students to have some programming skills.

QUESTIONS AND PROBLEMS

Problem 20.1.

Consider a position consisting of a $100,000 investment in asset A and a $100,000 investment in asset B. Assume that the daily volatilities of both assets are 1% and that the coefficient of correlation between their returns is 0.3. What is the 5-day 99% VaR for the portfolio?

The standard deviation of the daily change in the investment in each asset is $1,000. The variance of the portfolio's daily change is

$$1,000^2 + 1,000^2 + 2 \times 0.3 \times 1,000 \times 1,000 = 2,600,000$$

The standard deviation of the portfolio's daily change is the square root of this or $1,612.45. The standard deviation of the 5-day change is

$$1,612.45 \times \sqrt{5} = \$3,605.55$$

From the tables of $N(x)$ we see that $N(-2.33) = 0.01$. This means that 1% of a normal distribution lies more than 2.33 standard deviations below the mean. The 5-day 99 percent value at risk is therefore $2.33 \times 3,605.55 = \$8,401$.

Problem 20.2.

Describe three ways of handling interest-rate-dependent instruments when the model building approach is used to calculate VaR. How would you handle interest-rate-dependent instruments when historical simulation is used to calculate VaR?

The three alternative procedures mentioned in the chapter for handling interest rates when the model building approach is used to calculate VaR involve (a) the use of the duration model, (b) the use of cash flow mapping, and (c) the use of principal components analysis. When historical simulation is used we need to assume that the change in the zero-coupon yield curve between Day m and Day $m+1$ is the same as that between Day i and Day $i+1$ for different values of i. In the case of a LIBOR, the zero curve is usually calculated from deposit rates, Eurodollar futures quotes, and swap rates. We can assume that the percentage change in each of these between Day m and Day $m+1$ is the same as that between Day i and Day $i+1$. In the case of a Treasury curve it is usually calculated from the yields on Treasury instruments. Again we can assume that the percentage change in each of these between Day m and Day $m+1$ is the same as that between Day i and Day $i+1$.

Problem 20.3.

A financial institution owns a portfolio of options on the U.S. dollar–sterling exchange rate. The delta of the portfolio is 56.0. The current exchange rate is 1.5000. Derive an approximate linear relationship between the change in the portfolio value and the percentage change in the exchange rate. If the daily volatility of the exchange rate is 0.7%, estimate the 10-day 99% VaR.

The approximate relationship between the daily change in the portfolio value, ΔP, and the daily change in the exchange rate, ΔS, is

$$\Delta P = 56\Delta S$$

The percentage daily change in the exchange rate, Δx, equals $\Delta S/1.5$. It follows that

$$\Delta P = 56 \times 1.5\Delta x$$

or

$$\Delta P = 84\Delta x$$

The standard deviation of Δx equals the daily volatility of the exchange rate, or 0.7 percent. The standard deviation of ΔP is therefore $84 \times 0.007 = 0.588$. It follows that the 10-day 99 percent VaR for the portfolio is

$$0.588 \times 2.33 \times \sqrt{10} = 4.33$$

Problem 20.4.

Suppose you know that the gamma of the portfolio in the previous question is 16.2. How does this change your estimate of the relationship between the change in the portfolio value and the percentage change in the exchange rate?

The relationship is

$$\Delta P = 56 \times 1.5\Delta x + \frac{1}{2} \times 1.5^2 \times 16.2 \times \Delta x^2$$

or

$$\Delta P = 84\Delta x + 18.225\Delta x^2$$

Problem 20.5.

Suppose that the daily change in the value of a portfolio is, to a good approximation, linearly dependent on two factors, calculated from a principal components analysis. The delta of a portfolio with respect to the first factor is 6 and the delta with respect to the second factor is –4. The standard deviations of the factor are 20 and 8, respectively. What is the 5-day 90% VaR?

The factors calculated from a principal components analysis are uncorrelated. The daily variance of the portfolio is

$$6^2 \times 20^2 + 4^2 \times 8^2 = 15,424$$

and the daily standard deviation is $\sqrt{15,424} = \$124.19$. Since $N(-1.282) = 0.9$, the 5-day 90% value at risk is

$$124.19 \times \sqrt{5} \times 1.282 = \$356.01$$

Problem 20.6.

Suppose a company has a portfolio consisting of positions in stocks, bonds, foreign exchange, and commodities. Assume there are no derivatives. Explain the assumptions underlying (a) the linear model and (b) the historical simulation model for calculating VaR.

The linear model assumes that the percentage daily change in each market variable has a normal probability distribution. The historical simulation model assumes that the probability distribution observed for the percentage daily changes in the market variables in the past is the probability distribution that will apply over the next day.

Problem 20.7.

Explain how an interest rate swap is mapped into a portfolio of zero-coupon bonds with standard maturities for the purposes of a VaR calculation.

When a final exchange of principal is added in, the floating side is equivalent a zero coupon bond with a maturity date equal to the date of the next payment. The fixed side is a coupon-bearing bond, which is equivalent to a portfolio of zero-coupon bonds. The swap can therefore be mapped into a portfolio of zero-coupon bonds with maturity dates corresponding to the payment dates. Each of the zero-coupon bonds can then be mapped into positions in the adjacent standard-maturity zero-coupon bonds.

Problem 20.8.

Explain the difference between Value at Risk and Expected Shortfall.

Value at risk is the loss that is expected to be exceeded $(100 - X)\%$ of the time in N days for specified parameter values, X and N. Expected shortfall is the expected loss conditional that the loss is greater than the Value at Risk.

Problem 20.9.

Explain why the linear model can provide only approximate estimates of VaR for a portfolio containing options.

The change in the value of an option is not linearly related to the change in the value of the underlying variables. When the change in the values of underlying variables is normal, the change in the value of the option is non-normal. The linear model assumes that it is normal and is, therefore, only an approximation.

Problem 20.10.

Verify that the 0.3-year zero-coupon bond in the cash-flow mapping example in the appendix to this chapter is mapped into a \$37,397 position in a three-month bond and a \$11,793 position in a six-month bond.

The 0.3-year cash flow is mapped into a 3-month zero-coupon bond and a 6-month zero-coupon bond. The 0.25 and 0.50 year rates are 5.50 and 6.00 respectively. Linear interpolation gives the 0.30-year rate as 5.60%. The present value of \$50,000 received at time 0.3 years is

$$\frac{50,000}{1.056^{0.30}} = 49,189.32$$

The volatility of 0.25-year and 0.50-year zero-coupon bonds are 0.06% and 0.10% per day respectively. The interpolated volatility of a 0.30-year zero-coupon bond is therefore 0.068% per day.

Assume that α of the value of the 0.30-year cash flow gets allocated to a 3-month zero-coupon bond and $1 - \alpha$ to a six-month zero coupon bond. To match variances we must have

$$0.00068^2 = 0.0006^2\alpha^2 + 0.001^2(1-\alpha)^2 + 2 \times 0.9 \times 0.0006 \times 0.001\alpha(1-\alpha)$$

or

$$0.28\alpha^2 - 0.92\alpha + 0.5376 = 0$$

Using the formula for the solution to a quadratic equation

$$\alpha = \frac{-0.92 + \sqrt{0.92^2 - 4 \times 0.28 \times 0.5376}}{2 \times 0.28} = 0.760259$$

this means that a value of $0.760259 \times 49,189.32 = \$37,397$ is allocated to the three-month bond and a value of $0.239741 \times 49,189.32 = \$11,793$ is allocated to the six-month bond.

The 0.3-year cash flow is therefore equivalent to a position of \$37,397 in a 3-month zero-coupon bond and a position of \$11,793 in a 6-month zero-coupon bond. This is consistent with the results in Table 20A.2 of the appendix to Chapter 20.

Problem 20.11.

Suppose that the 5-year rate is 6%, the seven year rate is 7% (both expressed with annual compounding), the daily volatility of a 5-year zero-coupon bond is 0.5%, and the daily volatility of a 7-year zero-coupon bond is 0.58%. The correlation between daily returns on the two bonds is 0.6. Map a cash flow of \$1,000 received at time 6.5 years into a position in a five-year bond and a position in a seven-year bond using the approach in the appendix. What cash flows in five and seven years are equivalent to the 6.5-year cash flow?

The 6.5-year cash flow is mapped into a 5-year zero-coupon bond and a 7-year zero-coupon bond. The 5-year and 7-year rates are 6% and 7% respectively. Linear interpolation gives the 6.5-year rate as 6.75%. The present value of \$1,000 received at time 6.5 years is

$$\frac{1,000}{1.0675^{6.5}} = 654.05$$

The volatility of 5-year and 7-year zero-coupon bonds are 0.50% and 0.58% per day respectively. The interpolated volatility of a 6.5-year zero-coupon bond is therefore 0.56% per day.

Assume that α of the value of the 6.5-year cash flow gets allocated to a 5-year zero-coupon bond and $1 - \alpha$ to a 7-year zero coupon bond. To match variances we must have

$$.56^2 = .50^2\alpha^2 + .58^2(1-\alpha)^2 + 2 \times 0.6 \times .50 \times .58\alpha(1-\alpha)$$

or

$$.2384\alpha^2 - .3248\alpha + .0228 = 0$$

Using the formula for the solution to a quadratic equation

$$\alpha = \frac{.3248 - \sqrt{.3248^2 - 4 \times .2384 \times .0228}}{2 \times .2384} = 0.074243$$

this means that a value of $0.074243 \times 654.05 =$ \$48.56 is allocated to the 5-year bond and a value of $0.925757 \times 654.05 =$ \$605.49 is allocated to the 7-year bond. The 6.5-year cash flow is therefore equivalent to a position of \$48.56 in a 5-year zero-coupon bond and a position of \$605.49 in a 7-year zero-coupon bond.

The equivalent 5-year and 7-year cash flows are $48.56 \times 1.06^5 = 64.98$ and $605.49 \times 1.07^7 = 972.28$.

Problem 20.12.

Some time ago a company has entered into a forward contract to buy £1 million for \$1.5 million. The contract now has six months to maturity. The daily volatility of a six-month zero-coupon sterling bond (when its price is translated to dollars) is 0.06%

and the daily volatility of a six-month zero-coupon dollar bond is 0.05%. The correlation between returns from the two bonds is 0.8. The current exchange rate is 1.53. Calculate the standard deviation of the change in the dollar value of the forward contract in one day. What is the 10-day 99% VaR? Assume that the six-month interest rate in both sterling and dollars is 5% per annum with continuous compounding.

The contract is a long position in a sterling bond combined with a short position in a dollar bond. The value of the sterling bond is $1.53e^{-0.05\times0.5}$ or \$1.492 million. The value of the dollar bond is $1.5e^{-0.05\times0.5}$ or \$1.463 million. The variance of the change in the value of the contract in one day is

$$1.492^2 \times 0.0006^2 + 1.463^2 \times 0.0005^2 - 2 \times 0.8 \times 1.492 \times 0.0006 \times 1.463 \times 0.0005$$

$$= 0.000000288$$

The standard deviation is therefore \$0.000537 million. The 10-day 99% VaR is $0.000537 \times \sqrt{10} \times 2.33 =$ \$0.00396 million.

Problem 20.13.

The text calculates a VaR estimate for the example in Table 20.5 assuming two factors. How does the estimate change if you assume (a) one factor and (b) three factors.

If we assume only one factor, the model is

$$\Delta P = -0.08 f_1$$

The standard deviation of f_1 is 17.49. The standard deviation of ΔP is therefore $0.08 \times 17.49 = 1.40$ and the 1-day 99 percent value at risk is $1.40 \times 2.33 = 3.26$. If we assume three factors, our exposure to the third factor is

$$10 \times (-0.37) + 4 \times (-0.38) - 8 \times (-0.30) - 7 \times (-0.12) + 2 \times (-0.04) = -2.06$$

The model is therefore

$$\Delta P = -0.08 f_1 - 4.40 f_2 - 2.06 f_3$$

The variance of ΔP is

$$0.08^2 \times 17.49^2 + 4.40^2 \times 6.05^2 + 2.06^2 \times 3.10^2 = 751.36$$

The standard deviation of ΔP is $\sqrt{751.36} = 27.41$ and the 1-day 99% value at risk is $27.41 \times 2.33 =$ \$63.87.

The example illustrates that the relative importance of different factors depends on the portfolio being considered. Normally the second factor is less important than the first, but in this case it is much more important.

Problem 20.14.

A bank has a portfolio of options on an asset. The delta of the options is –30 and the gamma is –5. Explain how these numbers can be interpreted. The asset price is 20 and its volatility per day is 1%. Adapt Sample Application E in the DerivaGem Application Builder software to calcualte VaR.

The delta of the options is the rate if change of the value of the options with respect to the price of the asset. When the asset price increases by a small amount the value of the options decrease by 30 times this amount. The gamma of the options is the rate of change of their delta with respect to the price of the asset. When the asset price increases by a small amount, the delta of the portfolio decreases by five times this amount.

By entering 20 for S, 1% for the volatility per day, -30 for delta, -5 for gamma, and recomputing we see that $E(\Delta P) = -0.10$, $E(\Delta P^2) = 36.03$, and $E(\Delta P^3) = -32.415$. The 1-day, 99% VaR given by the software for the quadratic approximation is 14.5. This is a 99% 1-day VaR. The VaR is calculated using the formulas in footnote 9 and the results in Technical Note 10.

Problem 20.15.

Suppose that in Problem 20.14 the vega of the portfolio is -2 per 1% change in the annual volatility. Derive a model relating the change in the portfolio value in one day to delta, gamma, and vega. Explain without doing detailed calculations how you would use the model to calculate a VaR estimate.

Define σ as the volatility per year, $\Delta\sigma$ as the change in σ in one day, and Δw and the proportional change in σ in one day. We measure in σ as a multiple of 1% so that the current value of σ is $1 \times \sqrt{252} = 15.87$. The delta-gamma-vega model is

$$\Delta P = -30\Delta S - .5 \times 5 \times (\Delta S)^2 - 2\Delta\sigma$$

or

$$\Delta P = -30 \times 20\Delta x - 0.5 \times 5 \times 20^2(\Delta x)^2 - 2 \times 15.87\Delta w$$

which simplifies to

$$\Delta P = -600\Delta x - 1{,}000(\Delta x)^2 - 31.74\Delta w$$

The change in the portfolio value now depends on two market variables. Once the daily volatility of σ and the correlation between σ and S have been estimated we can estimate moments of ΔP and use a Cornish–Fisher expansion.

ASSIGNMENT QUESTIONS

Problem 20.16.

A company has a position in bonds worth \$6 million. The modified duration of the portfolio is 5.2 years. Assume that only parallel shifts in the yield curve can take place and that the standard deviation of the daily yield change (when yield is measured in percent) is 0.09. Use the duration model to estimate the 20-day 90% VaR for the portfolio. Explain *carefully the weaknesses of this approach to calculating VaR. Explain two alternatives that give more accuracy.*

The change in the value of the portfolio for a small change Δy in the yield is approximately $-DB\Delta y$ where D is the duration and B is the value of the portfolio. It follows that the standard deviation of the daily change in the value of the bond portfolio equals $DB\sigma_y$ where σ_y is the standard deviation of the daily change in the yield. In this case $D = 5.2$, $B = 6{,}000{,}000$, and $\sigma_y = 0.0009$ so that the standard deviation of the daily change in the value of the bond portfolio is

$$5.2 \times 6{,}000{,}000 \times 0.0009 = 28{,}080$$

The 20-day 90% VaR for the portfolio is $1.282 \times 28{,}080 \times \sqrt{20} = 160{,}990$ or \$160,990. This approach assumes that only parallel shifts in the term structure can take place. Equivalently it assumes that all rates are perfectly correlated or that only one factor drives term structure movements. Alternative more accurate approaches described in the chapter are (a) cash flow mapping and (b) a principal components analysis.

Problem 20.17.

Consider a position consisting of a \$300,000 investment in gold and a \$500,000 investment in silver. Suppose that the daily volatilities of these two assets are 1.8% and 1.2% respectively, and that the coefficient of correlation between their returns is 0.6. What is the 10-day 97.5% VaR for the portfolio? By how much does diversification reduce the VaR?

The variance of the portfolio (in thousands of dollars) is

$$0.018^2 \times 300^2 + 0.012^2 \times 500^2 + 2 \times 300 \times 500 \times 0.6 \times 0.018 \times 0.012 = 104.04$$

The standard deviation is $\sqrt{104.04} = 10.2$. Since $N(-1.96) = 0.025$, the 1-day 97.5% VaR is $10.2 \times 1.96 = 19.99$ and the 10-day 97.5% VaR is $\sqrt{10} \times 19.99 = 63.22$. The 10-day 97.5% VaR is therefore \$63,220. The 10-day 97.5% value at risk for the gold investment is $5{,}400 \times \sqrt{10} \times 1.96 = 33{,}470$. The 10-day 97.5% value at risk for the silver investment is $6{,}000 \times \sqrt{10} \times 1.96 = 37{,}188$. The diversification benefit is

$$33{,}470 + 37{,}188 - 63{,}220 = \$7{,}438$$

Problem 20.18.

Consider a portfolio of options on a single asset. Suppose that the delta of the portfolio is 12, the value of the asset is \$10, and the daily volatility of the asset is 2%. Estimate the 1-day 95% VaR for the portfolio from the delta. Suppose next that the gamma of the portfolio is −2.6. Derive a quadratic relationship between the change in the portfolio value and the percentage change in the underlying asset price in one day. How would you use this in a Monte Carlo simulation?

An approximate relationship between the daily change in the value of the portfolio, ΔP and the proportional daily change in the value of the asset Δx is

$$\Delta P = 10 \times 12\Delta x = 120\Delta x$$

The standard deviation of Δx is 0.02. It follows that the standard deviation of ΔP is 2.4. The 1-day 95% VaR is $2.4 \times 1.65 = \$3.96$. T The quadratic relationship is

$$\Delta P = 10 \times 12\Delta x + 0.5 \times 10^2 \times (-2.6)\Delta x^2$$

or

$$\Delta P = 120\Delta x - 130\Delta x^2$$

this could be used in conjunction with Monte Carlo simulation. We would sample values for Δx and use this equation to convert the Δx samples to ΔP samples.

Problem 20.19.

A company has a long position in a two-year bond and a three-year bond as well as a short position in a five-year bond. Each bond has a principal of \$100 and pays a 5% coupon annually. Calculate the company's exposure to the one-year, two-year, three-year, four-year, and five-year rates. Use the data in Tables 20.3 and 20.4 to calculate a 20 day 95% VaR on the assumption that rate changes are explained by (a) one factor, (b) two factors, and (c) three factors. Assume that the zero-coupon yield curve is flat at 5%.

The cash flows are as follows

Year	1	2	3	4	5
2-yr Bond	5	105	0	0	0
3-yr Bond	5	5	105	0	0
5-yr Bond	−5	−5	−5	−5	−105
Total	5	105	100	−5	−105
Present Value	4.756	95.008	86.071	−4.094	-81.774
Impact of 1 bp change	-0.0005	-0.0190	-0.0258	0.0016	0.0409

The duration relationship is used to calculate the last row of the table. When the one-year rate increases by one basis point, the value of the cash flow in year 1 decreases by $1 \times 0.0001 \times 4.756 = 0.0005$; when the two year rate increases by one basis point, the value of the cash flow in year 2 decreases by $2 \times 0.0001 \times 95.008 = 0.0190$; and so on.

The sensitivity to the first factor is

$$-0.0005 \times 0.32 - 0.0190 \times 0.35 - 0.0258 \times 0.36 + 0.0016 \times 0.36 + 0.0409 \times 0.36$$

or -0.00081. The sensitivity to the second factor is

$$-0.0005 \times (-0.32) - 0.0190 \times (-0.10) - 0.0258 \times 0.02 + 0.0016 \times 0.14 + 0.0409 \times 0.17$$

or 0.0087 The sensitivity to the third factor is

$$-0.0005 \times (-0.37) - 0.0190 \times (-0.38) - 0.0258 \times (-0.30) + 0.0016 \times (-0.12)$$

$$+0.0409 \times (-0.04)$$

or 0.0133.

Assuming one factor, the standard deviation of the one-day change in the portfolio value is $0.00081 \times 17.49 = 0.0142$. The 20-day 95% VaR is therefore $0.0142 \times 1.645\sqrt{20} = 0.104$

Assuming two factors, the standard deviation of the one-day change in the portfolio value is

$$\sqrt{0.00081^2 \times 17.49^2 + 0.0087^2 \times 6.05^2} = 0.0545$$

The 20-day 95% VaR is therefore $0.0545 \times 1.645\sqrt{20} = 0.401$

Assuming three factors, the standard deviation of the one-day change in the portfolio value is

$$\sqrt{0.00081^2 \times 17.49^2 + 0.0087^2 \times 6.05^2 + 0.0133^2 \times 3.10^2} = 0.0683$$

The 20-day 95% VaR is therefore $0.0683 \times 1.645\sqrt{20} = 0.502$.

In this case the second and third factor are important in calculating VaR.

Problem 20.20.

A bank has written a call option on one stock and a put option on another stock. For the first option the stock price is 50, the strike price is 51, the volatility is 28% per annum, and the time to maturity is nine months. For the second option the stock price is 20, the strike price is 19, the volatility is 25% per annum, and the time to maturity is one year. Neither stock pays a dividend, the risk-free rate is 6% per annum, and the correlation between stock price returns is 0.4. Calculate a 10-day 99% VaR

a. *Using only deltas.*
b. *Using the partial simulation approach.*
c. *Using the full simulation approach.*

This assignment is useful for consolidating students' understanding of alternative approaches to calculating VaR, but it is calculation intensive. Realistically students need some programming skills to make the assignment feasible. My answer follows the usual practice of assuming that the 10-day 99% value at risk is $\sqrt{10}$ times the 1-day 99% value at risk. Some students may try to calculate a 10-day VaR directly, which is fine.

(a) From DerivaGem, the values of the two option positions are –5.413 and –1.014. The deltas are –0.589 and 0.284, respectively. An approximate linear model relating the change in the portfolio value to proportional change, Δx_1, in the first stock price and the proportional change, Δx_2, in the second stock price is

$$\Delta P = -0.589 \times 50\Delta x_1 + 0.284 \times 20\Delta x_2$$

or

$$\Delta P = -29.45\Delta x_1 + 5.68\Delta x_2$$

The daily volatility of the two stocks are $0.28/\sqrt{252} = 0.0176$ and $0.25/\sqrt{252} = 0.0157$, respectively. The one-day variance of ΔP is

$$29.45^2 \times 0.0176^2 + 5.68^2 \times 0.0157^2 - 2 \times 29.45 \times 0.0176 \times 5.68 \times 0.0157 \times 0.4 = 0.2396$$

The one day standard deviation is, therefore, 0.4895 and the 10-day 99% VaR is $2.33 \times \sqrt{10} \times 0.4895 = 3.61$.

(b) In the partial simulation approach, we simulate changes in the stock prices over a one-day period (building in the correlation) and then use the quadratic approximation to calculate the change in the portfolio value on each simulation trial. The one percentile point of the probability distribution of portfolio value changes turns out to be 1.22. The 10-day 99% value at risk is, therefore, $1.22\sqrt{10}$ or about 3.86.

(c) In the full simulation approach, we simulate changes in the stock price over one-day (building in the correlation) and revalue the portfolio on each simulation trial. The results are very similar to (b) and the estimate of the 10-day 99% value at risk is about 3.86.

Problem 20.21.

A common complaint of risk managers is that the model building approach (either linear or quadratic) does not work well when delta is close to zero. Test what happens when delta is close to zero in using Sample Application E in the DerivaGem Application Builder software. (You can do this by experimenting with different option positions and adjusting the position in the underlying to give a delta of zero.) Explain the results you get.

We can create a portfolio with zero delta in Sample Application E by changing the position in the stock from 1,000 to 513.58. (This reduces delta by $1,000 - 513.58 = 486.42$.) In this case the true VaR is 48.86; the VaR given by the linear model is 0.00; and the VaR given by the quadratic model is -35.71.

Other zero-delta examples can be created by changing the option portfolio and then zeroing out delta by adjusting the position in the underlying asset. The results are similar. The software shows that neither the linear model nor the quadratic model gives good answers when delta is zero. The linear model always gives a VaR of zero because the model assumes that the portfolio has no risk. (For example, in the case of one underlying asset $\Delta P = \Delta \Delta S$.) The quadratic model gives a negative VaR because ΔP is always positive in this model. ($\Delta P = 0.5\Gamma(\Delta S)^2$.

In practice many portfolios do have deltas close to zero because of the hedging activities described in Chapter 17. This has led many financial institutions to prefer historical simulation to the model building approach.

CHAPTER 21

Estimating Volatilities and Correlations

Notes for the Instructor

This chapter covers exponentially weighted moving average (EWMA) and GARCH (1,1) procedures for estimating the current level of a volatility or correlation. It explains maximum likelihood methods.

At the outset is important to make sure students understand the notation. The variable σ_n is the volatility estimated for day n *at the end of* day $n-1$; u_n is the realized return *during* day n. The EWMA approach, although not as sophisticated as GARCH(1,1), is widely used and is a useful lead-in to GARCH(1,1). After explaining both EWMA and GARCH (1,1) the chapter discusses maximum likelihood methods, the use of GARCH (1,1) for forecasting and the calculation of vega, and the application of the ideas to correlations.

Although there is readily available software for implementing GARCH (1,1) I like students to develop their own Excel applications. By doing this they develop a much better understanding of how maximum likelihood methods work. As indicated in the text the Solver routine in Excel is very effective if used in such a way that all the parameters being searched for are the same order of magnitude.

An interesting case study to teach in conjunction with the material in this chapter is Philippe Jorion's one on Orange County. See

http://www.gsm.uci.edu/~jorion/oc/case.html

Problems 21.15 to 21.18 can be used as assignment questions. They vary quite a bit in terms of the amount of time likely to be required. Problems 21.15 and 21.16 are a fairly quick test of whether students understand how EWMA and GARCH work. Problem 21.18 is somewhat longer. Problem 21.17 is longer again and requires some Excel skills.

QUESTIONS AND PROBLEMS.

Problem 21.1.

Explain the exponentially weighted moving average (EWMA) model for estimating volatility from historical data.

Define u_i as $(S_i - S_{i-1})/S_{i-1}$, where S_i is value of a market variable on day i. In the EWMA model, the variance rate of the market variable (i.e., the square of its volatility) calculated for day n is a weighted average of the u_{n-i}^2's ($i = 1, 2, 3, \ldots$). For some constant λ ($0 < \lambda < 1$) the weight given to u_{n-i-1}^2 is λ times the weight given to u_{n-i}^2. The volatility estimated for day n, σ_n, is related to the volatility estimated for day $n-1$, σ_{n-1}, by

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2$$

This formula shows that the EWMA model has one very attractive property. To calculate the volatility estimate for day n, it is sufficient to know the volatility estimate for day $n-1$ and u_{n-1}.

Problem 21.2.

What is the difference between the exponentially weighted moving average model and the GARCH(1,1) model for updating volatilities?

The EWMA model produces a forecast of the daily variance rate for day n which is a weighted average of (i) the forecast for day $n-1$, and (ii) the square of the proportional change on day $n-1$. The GARCH (1,1) model produces a forecast of the daily variance for day n which is a weighted average of (i) the forecast for day $n-1$, (ii) the square of the proportional change on day $n-1$. and (iii) a long run average variance rate. GARCH (1,1) adapts the EWMA model by giving some weight to a long run average variance rate. Whereas the EWMA has no mean reversion, GARCH (1,1) is consistent with a mean-reverting variance rate model.

Problem 21.3.

The most recent estimate of the daily volatility of an asset is 1.5% and the price of the asset at the close of trading yesterday was \$30.00. The parameter λ in the EWMA model is 0.94. Suppose that the price of the asset at the close of trading today is \$30.50. How will this cause the volatility to be updated by the EWMA model?

In this case $\sigma_{n-1} = 0.015$ and $u_n = 0.5/30 = 0.01667$, so that equation (21.7) gives

$$\sigma_n^2 = 0.94 \times 0.015^2 + 0.06 \times 0.01667^2 = 0.0002281$$

The volatility estimate on day n is therefore $\sqrt{0.0002281} = 0.015103$ or 1.5103%.

Problem 21.4.

A company uses an EWMA model for forecasting volatility. It decides to change the parameter λ from 0.95 to 0.85. Explain the likely impact on the forecasts.

Reducing λ from 0.95 to 0.85 means that more weight is put on recent observations of u_i^2 and less weight is given to older observations. Volatilities calculated with $\lambda = 0.85$ will react more quickly to new information and will "bounce around" much more than volatilities calculated with $\lambda = 0.95$.

Problem 21.5.

The volatility of a certain market variable is 30% per annum. Calculate a 99% confidence interval for the size of the percentage daily change in the variable.

The volatility per day is $30/\sqrt{252} = 1.89\%$. There is a 99% chance that a normally distributed variable will lie within 2.57 standard deviations. We are therefore 99% confident that the daily change will be less than $2.57 \times 1.89 = 4.86\%$.

Problem 21.6.

A company uses the GARCH(1,1) model for updating volatility. The three parameters are ω, α, and β. Describe the impact of making a small increase in each of the parameters while keeping the others fixed.

The weight given to the long-run average variance rate is $1 - \alpha - \beta$ and the long-run average variance rate is $\omega/(1 - \alpha - \beta)$. Increasing ω increases the long-run average variance rate; Increasing α increases the weight given to the most recent data item, reduces the weight given to the long-run average variance rate, and increases the level of the long-run average variance rate. Increasing β increases the weight given to the previous variance estimate, reduces the weight given to the long-run average variance rate, and increases the level of the long-run average variance rate.

Problem 21.7.

The most recent estimate of the daily volatility of the U.S. dollar–sterling exchange rate is 0.6% and the exchange rate at 4 p.m. yesterday was 1.5000. The parameter λ in the EWMA model is 0.9. Suppose that the exchange rate at 4 p.m. today proves to be 1.4950. How would the estimate of the daily volatility be updated?

The proportional daily change is $-0.005/1.5000 = -0.003333$. The current daily variance estimate is $0.006^2 = 0.000036$. The new daily variance estimate is

$$0.9 \times 0.000036 + 0.1 \times 0.003333^2 = 0.000033511$$

The new volatility is the square root of this. It is 0.00579 or 0.579%.

Problem 21.8.

Assume that S&P 500 at close of trading yesterday was 1,040 and the daily volatility of the index was estimated as 1% per day at that time. The parameters in a GARCH(1,1) model are $\omega = 0.000002$, $\alpha = 0.06$, and $\beta = 0.92$. If the level of the index at close of trading today is 1,060, what is the new volatility estimate?

With the usual notation $u_{n-1} = 20/1040 = 0.01923$ so that

$$\sigma_n^2 = 0.000002 + 0.06 \times 0.01923^2 + 0.92 \times 0.01^2 = 0.0001162$$

so that $\sigma_n = 0.01078$. The new volatility estimate is therefore 1.078% per day.

Problem 21.9.

Suppose that the daily volatilities of asset A and asset B calculated at the close of trading yesterday are 1.6% and 2.5%, respectively. The prices of the assets at close of trading yesterday were \$20 and \$40 and the estimate of the coefficient of correlation between the returns on the two assets was 0.25. The parameter λ used in the EWMA model is 0.95.

(a) Calculate the current estimate of the covariance between the assets.

(b) On the assumption that the prices of the assets at close of trading today are \$20.5 and \$40.5, update the correlation estimate.

(a) The volatilities and correlation imply that the current estimate of the covariance is $0.25 \times 0.016 \times 0.025 = 0.0001$.

(b) If the prices of the assets at close of trading are \$20.5 and \$40.5, the proportional changes are $0.5/20 = 0.025$ and $0.5/40 = 0.0125$. The new covariance estimate is

$$0.95 \times 0.0001 + 0.05 \times 0.025 \times 0.0125 = 0.0001106$$

The new variance estimate for asset A is

$$0.95 \times 0.016^2 + 0.05 \times 0.025^2 = 0.00027445$$

so that the new volatility is 0.0166. The new variance estimate for asset B is

$$0.95 \times 0.025^2 + 0.05 \times 0.0125^2 = 0.000601562$$

so that the new volatility is 0.0245. The new correlation estimate is

$$\frac{0.0001106}{0.0166 \times 0.0245} = 0.272$$

Problem 21.10.

The parameters of a GARCH(1,1) model are estimated as $\omega = 0.000004$, $\alpha = 0.05$, and $\beta = 0.92$. What is the long-run average volatility and what is the equation describing the way that the variance rate reverts to its long-run average? If the current volatility is 20% per year, what is the expected volatility in 20 days?

The long-run average variance rate is $\omega/(1 - \alpha - \beta)$ or $0.000004/0.03 = 0.0001333$. The long-run average volatility is $\sqrt{0.0001333}$ or 1.155%. The equation describing the way the variance rate reverts to its long-run average is equation (21.13)

$$E[\sigma_{n+k}^2] = V_L + (\alpha + \beta)^k(\sigma_n^2 - V_L)$$

In this case

$$E[\sigma_{n+k}^2] = 0.0001333 + 0.97^k(\sigma_n^2 - 0.0001333)$$

If the current volatility is 20% per year, $\sigma_n = 0.2/\sqrt{252} = 0.0126$. The expected variance rate in 20 days is

$$0.0001333 + 0.97^{20}(0.0126^2 - 0.0001333) = 0.0001471$$

The expected volatility in 20 days is therefore $\sqrt{0.0001471} = 0.0121$ or 1.21% per day.

Problem 21.11.

Suppose that the current daily volatilities of asset X and asset Y are 1.0% and 1.2%, respectively. The prices of the assets at close of trading yesterday were \$30 and \$50 and the estimate of the coefficient of correlation between the returns on the two assets made at *this time was 0.50. Correlations and volatilities are updated using a GARCH(1,1) model. The estimates of the model's parameters are $\alpha = 0.04$ and $\beta = 0.94$. For the correlation $\omega = 0.000001$ and for the volatilities $\omega = 0.000003$. If the prices of the two assets at close of trading today are \$31 and \$51, how is the correlation estimate updated?*

Using the notation in the text $\sigma_{u,n-1} = 0.01$ and $\sigma_{v,n-1} = 0.012$ and the most recent estimate of the covariance between the asset returns is $\text{cov}_{n-1} = 0.01 \times 0.012 \times 0.50 = 0.00006$. The variable $u_{n-1} = 1/30 = 0.03333$ and the variable $v_{n-1} = 1/50 = 0.02$. The new estimate of the covariance, cov_n, is

$$0.000001 + 0.04 \times 0.03333 \times 0.02 + 0.94 \times 0.00006 = 0.0000841$$

The new estimate of the variance of the first asset, $\sigma_{u,n}^2$ is

$$0.000003 + 0.04 \times 0.03333^2 + 0.94 \times 0.01^2 = 0.0001414$$

so that $\sigma_{u,n} = \sqrt{0.0001414} = 0.01189$ or 1.189%. The new estimate of the variance of the second asset, $\sigma_{v,n}^2$ is

$$0.000003 + 0.04 \times 0.02^2 + 0.94 \times 0.012^2 = 0.0001544$$

so that $\sigma_{v,n} = \sqrt{0.0001544} = 0.01242$ or 1.242%. The new estimate of the correlation between the assets is therefore $0.0000841/(0.01189 \times 0.01242) = 0.569$.

Problem 21.12.

Suppose that the daily volatility of the FT-SE 100 stock index (measured in pounds sterling) is 1.8% and the daily volatility of the dollar/sterling exchange rate is 0.9%. Suppose further that the correlation between the FT-SE 100 and the dollar/sterling exchange rate is 0.4. What is the volatility of the FT-SE 100 when it is translated to U.S. dollars? Assume that the dollar/sterling exchange rate is expressed as the number of U.S. dollars per pound sterling. (Hint: When $Z = XY$, the percentage daily change in Z is approximately equal to the percentage daily change in X plus the percentage daily change in Y.)

The FT-SE expressed in dollars is XY where X is the FT-SE expressed in sterling and Y is the exchange rate (value of one pound in dollars). Define x_i as the proportional change in X on day i and y_i as the proportional change in Y on day i. The proportional change in XY is approximately $x_i + y_i$. The standard deviation of x_i is 0.018 and the standard deviation of y_i is 0.009. The correlation between the two is 0.4. The variance of $x_i + y_i$ is therefore

$$0.018^2 + 0.009^2 + 2 \times 0.018 \times 0.009 \times 0.4 = 0.0005346$$

so that the volatility of $x_i + y_i$ is 0.0231 or 2.31%. This is the volatility of the FT-SE expressed in dollars. Note that it is greater than the volatility of the FT-SE expressed in sterling. This is the impact of the positive correlation. When the FT-SE increases the value of sterling measured in dollars also tends to increase. This creates an even bigger increase in the value of FT-SE measured in dollars. Similarly for a decrease in the FT-SE.

Problem 21.13.

Suppose that in Problem 21.12 the correlation between the S&P 500 Index (measured in dollars) and the FT-SE 100 Index (measured in sterling) is 0.7, the correlation between the S&P 500 index (measured in dollars) and the dollar-sterling exchange rate is 0.3, and the daily volatility of the S&P 500 Index is 1.6%. What is the correlation between the S&P 500 Index (measured in dollars) and the FT-SE 100 Index when it is translated to dollars? (Hint: For three variables X, Y, and Z, the covariance between $X+Y$ and Z equals the covariance between X and Z plus the covariance between Y and Z.)

Continuing with the notation in Problem 21.12, define z_i as the proportional change in the value of the S&P 500 on day i. The covariance between x_i and z_i is $0.7 \times 0.018 \times 0.016 = 0.0002016$. The covariance between y_i and z_i is $0.3 \times 0.009 \times 0.016 = 0.0000432$. The covariance between $x_i + y_i$ and z_i equals the covariance between x_i and z_i plus the covariance between y_i and z_i. It is

$$0.0002016 + 0.0000432 = 0.0002448$$

The correlation between $x_i + y_i$ and z_i is

$$\frac{0.0002448}{0.016 \times 0.0231} = 0.662$$

Note that the volatility of the S&P 500 drops out in this calculation.

Problem 21.14.

Show that the GARCH (1,1) model

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

in equation (21.9) is equivalent to the stochastic volatility model

$$dV = a(V_L - V)\,dt + \xi V\,dz$$

where time is measured in days and V is the square of the volatility of the asset price and

$$a = 1 - \alpha - \beta$$

$$V_L = \frac{\omega}{1 - \alpha - \beta}$$

$$\xi = \alpha\sqrt{2}$$

What is the stochastic volatility model when time is measure in years?

(Hint: The variable u_{n-1} is the return on the asset price in time Δt. It can be assumed to be normally distributed with mean zero and standard deviation σ_{n-1}. It follows that the mean of u_{n-1}^2 and u_{n-1}^4 are σ_{n-1}^2 and $3\sigma_{n-1}^4$, respectively.)

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

so that

$$\sigma_n^2 - \sigma_{n-1}^2 = \omega + (\beta - 1)\sigma_{n-1}^2 + \alpha u_{n-1}^2$$

The variable u_{n-1}^2 has a mean of σ_{n-1}^2 and a variance of

$$E(u_{n-1})^4 - [E(u_{n-1}^2)]^2 = 2\sigma_{n-1}^4$$

The standard deviation of u_{n-1}^2 is $\sqrt{2}\sigma_{n-1}^2$. Assuming the u_i are generated by a Wiener process, dz, we can therefore write

$$u_{n-1}^2 = \sigma_{n-1}^2 + \sqrt{2}\sigma_{n-1}^2 \epsilon$$

where ϵ is a random sample from a standard normal distribution. Substituting this into the equation for $\sigma_n^2 - \sigma_{n-1}^2$ we get

$$\sigma_n^2 - \sigma_{n-1}^2 = \omega + (\alpha + \beta - 1)\sigma_{n-1}^2 + \alpha\sqrt{2}\sigma_{n-1}^2 \epsilon$$

We can write $\Delta V = \sigma_n^2 - \sigma_{n-1}^2$ and $V = \sigma_{n-1}^2$. Also $a = 1 - \alpha - \beta$, $aV_L = \omega$, and $\xi = \alpha\sqrt{2}$ so that

$$\Delta V = a(V_L - V) + \xi\epsilon V$$

Because time is measured in days, $\Delta t = 1$ and

$$\Delta V = a(V_L - V)\Delta t + \xi V \epsilon \sqrt{\Delta t}$$

The result follows.

When time is measured in years $\Delta t = 1/252$ so that

$$\Delta V = a(V_L - V)252\Delta t + \xi V \epsilon \sqrt{252}\sqrt{\Delta t}$$

and the process for V is

$$dV = 252a(V_L - V)\,dt + \xi V \sqrt{252}\,dz$$

ASSIGNMENT QUESTIONS

Problem 21.15.

Suppose that the price of gold at close of trading yesterday was \$600 and its volatility was estimated as 1.3% per day. The price at the close of trading today is \$596. Update the volatility estimate using

(a) The EWMA model with $\lambda = 0.94$

(b) The GARCH(1,1) model with $\omega = 0.000002$, $\alpha = 0.04$, *and* $\beta = 0.94$.

The proportional change in the price of gold is $-4/600 = -0.00667$. Using the EWMA model the variance is updated to

$$0.94 \times 0.013^2 + 0.06 \times 0.00667^2 = 0.00016153$$

so that the new daily volatility is $\sqrt{0.00016153} = 0.01271$ or 1.271% per day. Using GARCH (1,1) the variance is updated to

$$0.000002 + 0.94 \times 0.013^2 + 0.04 \times 0.00667^2 = 0.00016264$$

so that the new daily volatility is $\sqrt{0.00016264} = 0.1275$ or 1.275% per day.

Problem 21.16.

Suppose that in Problem 21.15 the price of silver at the close of trading yesterday was \$16, its volatility was estimated as 1.5% per day, and its correlation with gold was estimated as 0.8. The price of silver at the close of trading today is unchanged at \$16. Update the volatility of silver and the correlation between silver and gold using the two models in Problem 21.15. In practice, is the ω *parameter likely to be the same for gold and silver?*

The proportional change in the price of silver is zero. Using the EWMA model the variance is updated to

$$0.94 \times 0.015^2 + 0.06 \times 0 = 0.0002115$$

so that the new daily volatility is $\sqrt{0.0002115} = 0.01454$ or 1.454% per day. Using GARCH (1,1) the variance is updated to

$$0.000002 + 0.94 \times 0.015^2 + 0.04 \times 0 = 0.0002135$$

so that the new daily volatility is $\sqrt{0.0002135} = 0.01461$ or 1.461% per day. The initial covariance is $0.8 \times 0.013 \times 0.015 = 0.000156$ Using EWMA the covariance is updated to

$$0.94 \times 0.000156 + 0.06 \times 0 = 0.00014664$$

so that the new correlation is $0.00014664/(0.01454 \times 0.01271) = 0.7934$ Using GARCH (1,1) the covariance is updated to

$$0.000002 + 0.94 \times 0.000156 + 0.04 \times 0 = 0.00014864$$

so that the new correlation is $0.00014864/(0.01461 \times 0.01275) = 0.7977$.

For a given α and β, the ω parameter defines the long run average value of a variance or a covariance. There is no reason why we should expect the long run average daily variance for gold and silver should be the same. There is also no reason why we should expect the long run average covariance between gold and silver to be the same as the long run average variance of gold or the long run average variance of silver. In practice, therefore, we are likely to want to allow ω in a GARCH(1,1) model to vary from market variable to market variable. (Some instructors may want to use this problem as a lead in to multivariate GARCH models.

Problem 21.17.

An Excel spreadsheet containing over 900 days of daily data on a number of different exchange rates and stock indices can be downloaded from the author's website: http://www.rotman.utoronto.ca/ hull. Choose one exchange rate and one stock index. Estimate the value of λ in the EWMA model that minimizes the value of

$$\sum_i (v_i - \beta_i)^2$$

where v_i is the variance forecast made at the end of day $i-1$ and β_i is the variance calculated from data between day i and day $i+25$. Use the Solver tool in Excel. Set the variance forecast at the end of the first day equal to the square of the return on that day to start the EWMA calculations.

My results give "best" values for λ higher than the 0.94 used by RiskMetrics. For AUD, BEF, CHF, DEM, DKK, ESP, FRF, GBP, ITL, NLG, and SEK they are 0.983, 0.967, 0.968, 0.960, 0.971, 0.983, 0.965, 0.977, 0.939, 0.962, and 0.989, respectively. For TSE, S&P, FTSE, CAC, and Nikkei, they are 0.991, 0.989, 0.958, 0.974, and 0.961, respectively.

Problem 21.18.

Suppose that the parameters in a GARCH (1,1) model are $\alpha = 0.03$, $\beta = 0.95$ and $\omega = 0.000002$.

a. *What is the long-run average volatility?*
b. *If the current volatility is 1.5% per day, what is your estimate of the volatility in 20, 40, and 60 days?*
c. *What volatility should be used to price 20-, 40-, and 60-day options?*
d. *Suppose that there is an event that increases the current volatility by 0.5% to 2% per day. Estimate the effect on the volatility in 20, 40, and 60 days.*
e. *Estimate by how much does the event increase the volatilities used to price 20-, 40-, and 60-day options?*

(a) The long-run average variance, V_L, is

$$\frac{\omega}{1-\alpha-\beta} = \frac{0.000002}{0.02} = 0.0001$$

The long run average volatility is $\sqrt{0.0001} = 0.01$ or 1% per day

(b) From equation (21.13) the expected variance in 20 days is

$$0.0001 + 0.98^{20}(0.015^2 - 0.0001) = 0.000183$$

The expected volatility per day is therefore $\sqrt{0.000183} = 0.0135$ or 1.35%. Similarly the expected volatilities in 40 and 60 days are 1.25% and 1.17%, respectively.

(c) In equation (21.14) $a = \ln(1/0.98) = 0.0202$ the variance used to price options in 20 days is

$$252(0.00001 + \frac{1 - e^{-0.0202\times20}}{0.0202 \times 20}(0.01566 - 0.00001) = 0.051$$

so that the volatility per annum is 22.61%. Similarly, the volatilities that should be used for 40- and 60-day options are 21.63% and 20.85% per annum, respectively.

(d) From equation (21.13) the expected variance in 20 days is

$$0.0001 + 0.98^{20}(0.02^2 - 0.0001) = 0.0003$$

The expected volatility per day is therefore $\sqrt{0.0003} = 0.0135$ or 1.73%. Similarly the expected volatilities in 40 and 60 days are 1.53% and 1.38% per day, respectively.

(e) When today's volatility increases from 1.5% per day (23.81% per year) to 2% per day (31.75% per year) the equation (21.15) gives the 20-day volatility increase as

$$\frac{1 - e^{-0.0202\times20}}{0.0202 \times 20} \times \frac{23.81}{22.61} \times (31.75 - 23.81) = 6.88$$

or 6.88% bringing the volatility up to 29.49%. Similarly the 40- and 60-day volatilities increase to 27.37% and 25.70%. A more exact calculation using equation (21.14) gives 29.56%, 27.76%, and 26.27% as the three volatilities.

CHAPTER 22
Credit Risk

Notes for the Instructor

This chapter covers the quantification of credit risk and prepares the way for the material in Chapter 23 on credit derivatives. It has been updated and improved for the seventh edition. After explaining credit ratings I spend some time talking about the difference between real world (physical) default probabilities and risk-neutral (implied) default probabilities. As Tables 22.4 and 22.5 show the difference between the two is much higher than might be expected with the risk-neutral default probabilities being higher. However, the extra expected return of bond traders as a result of this is not excessive. The important point to emphasize is that bonds do not default independently of each other and, as a result, there is systematic risk that is priced in the market.

Section 22.6 provides an outline of Merton's model for implying default probabilities from equity prices and how it is used in practice. Section 22.7 outlines how derivatives transactions should be adjusted for credit risk. Section 22.8 discusses netting, collateralization agreements and downgrade triggers. Section 22.9 introduces the Gaussian copula model which is used in both Basel II (see Business Snapshot 22.2) and in the valuation of credit derivatives (see Chapter 23).

I generally allow two classes for this chapter and two classes for Chapter 23. Problem 22.30 works well for class discussion. Problems 22.28, 22.29, 22.31, and 22.32 work well as assignments. Some Excel skills are necessary for Problems 22.28 and 22.32.

QUESTIONS AND PROBLEMS

Problem 22.1.

The spread between the yield on a three-year corporate bond and the yield on a similar risk-free bond is 50 basis points. The recovery rate is 30%. Estimate the average default intensity per year over the three-year period.

From equation (22.2) the average default intensity over the three years is $0.0050/(1-0.3) = 0.0071$ or 0.71% per year.

Problem 22.2.

Suppose that in Problem 22.1 the spread between the yield on a five-year bond issued by the same company and the yield on a similar risk-free bond is 60 basis points. Assume the same recovery rate of 30%. Estimate the average default intensity per year over the five-year period. What do your results indicate about the average default intensity in years 4 and 5?

From equation (22.2) the average default intensity over the five years is $0.0060/(1 - 0.3) = 0.0086$ or 0.86% per year. Using the results in the previous question, the default intensity is 0.71% per year for the first three years and

$$\frac{0.0086 \times 5 - 0.0071 \times 3}{2} = 0.0107$$

or 1.07% per year in years 4 and 5.

Problem 22.3.

Should researchers use real-world or risk-neutral default probabilities for a) calculating credit value at risk and b) adjusting the price of a derivative for defaults?

Real-world probabilities of default should be used for calculating credit value at risk. Risk-neutral probabilities of default should be used for adjusting the price of a derivative for default.

Problem 22.4.

How are recovery rates usually defined?

The recovery rate for a bond is the value of the bond immediately after the issuer defaults as a percent of its face value.

Problem 22.5.

Explain the difference between an unconditional default probability density and a default intensity.

The default intensity, $h(t)$ at time t is defined so that $h(t)\Delta t$ is the probability of default between times t and $t + \Delta t$ conditional on no default prior to time t. The unconditional default probability density $q(t)$ is defined so that $q(t)\Delta t$ is the probability of default between times t and $t + \Delta t$ as seen at time zero.

Problem 22.6.

Verify a) that the numbers in the second column of Table 22.4 are consistent with the numbers in Table 22.1 and b) that the numbers in the fourth column of Table 22.5 are consistent with the numbers in Table 22.4 and a recovery rate of 40%.

The first number in the second column of Table 22.4 is calculated as

$$-\frac{1}{7}\ln(1 - 0.00251) = 0.000359$$

or 0.04% per year. Other numbers in the column are calculated similarly. The numbers in the fourth column of Table 22.5 are the numbers in the second column of Table 22.4 multiplied by one minus the expected recovery rate. In this case the expected recovery rate is 0.4.

Problem 22.7.

Describe how netting works. A bank already has one transaction with a counterparty on its books. Explain why a new transaction by a bank with a counterparty can have the effect of increasing or reducing the bank's credit exposure to the counterparty.

Suppose company A goes bankrupt when it has a number of outstanding contracts with company B. Netting means that the contracts with a positive value to A are netted against those with a negative value in order to determine how much, if anything, company A owes company B. Company A is not allowed to "cherry pick" by keeping the positive-value contracts and defaulting on the negative-value contracts.

The new transaction will increase the bank's exposure to the counterparty if the contract tends to have a positive value whenever the existing contract has a positive value and a negative value whenever the existing contract has a negative value. However, if the new transaction tends to offset the existing transaction, it is likely to have the incremental effect of reducing credit risk.

Problem 22.8.

Suppose that the measure $\beta_{AB}(T)$ in equation (22.9) is the same in the real world and the risk-neutral world. Is the same true of the Gaussian copula measure, ρ_{AB}?

Equation (22.14) gives the relationship between $\beta_{AB}(T)$ and ρ_{AB}. This involves $Q_A(T)$ and $Q_B(T)$. These change as we move from the real world to the risk-neutral world. It follows that the relationship between $\beta_{AB}(T)$ and ρ_{AB} in the real world is not the same as in the risk-neutral world. If $\beta_{AB}(T)$ is the same in the two worlds, ρ_{AB} is not.

Problem 22.9.

What is meant by a haircut in a collateralization agreement. A company offers to post its own equity as collateral. How would you respond?

When securities are pledged as collateral the haircut is the discount applied to their market value for margin calculations. A company's own equity would not be good collateral. When the company defaults on its contracts its equity is likely to be worth very little.

Problem 22.10.

Explain the difference between the Gaussian copula model for the time to default and CreditMetrics as far as the following are concerned: a) the definition of a credit loss and b) the way in which default correlation is modeled.

(a) In the Gaussian copula model for time to default a credit loss is recognized only when a default occurs. In CreditMetrics it is recognized when there is a credit downgrade as well as when there is a default.

(b) In the Gaussian copula model of time to default, the default correlation arises because the value of the factor M. This defines the default environment or average default rate in the economy. In CreditMetrics a copula model is applied to credit ratings migration and this determines the joint probability of particular changes in the credit ratings of two companies.

Problem 22.11.

Suppose that the probability of company A defaulting during a two year period is 0.2 and the probability of company B defaulting during this period is 0.15. If the Gaussian copula measure of default correlation is 0.3, what is the binomial correlation measure?

In equation (22.14), $Q_A(2) = 0.2$, $Q_B(2) = 0.15$, and $\rho_{AB} = 0.3$. Also

$$x_A(2) = N^{-1}(0.2) = -0.84162$$

$$x_B(2) = N^{-1}(0.15) = -1.03643$$

$$M(-0.84162, -1.03643, 0.3) = 0.0522$$

$$\beta_{AB}(2) = \frac{0.0522 - 0.2 \times 0.15}{\sqrt{(0.2 - 0.2^2)(0.15 - 0.15^2)}} = 0.156$$

Problem 22.12.

Suppose that the LIBOR/swap curve is flat at 6% with continuous compounding and a five-year bond with a coupon of 5% (paid semiannually) sells for 90.00. How would an asset swap on the bond be structured? What is the asset swap spread that would be calculated in this situation?

Suppose that the principal is \$100. The asset swap is structured so that the \$10 is paid initially. After that \$2.50 is paid every six months. In return LIBOR plus a spread is received on the principal of \$100. The present value of the fixed payments is

$$10 + 2.5e^{-0.06\times0.5} + 2.5e^{-0.06\times1} + \ldots + 2.5e^{-0.06\times5} + 100e^{-0.06\times5} = 105.3579$$

The spread over LIBOR must therefore have a present value of 5.3579. The present value of \$1 received every six months for five years is 8.5105. The spread received every six months must therefore be 5.3579/8.5105 = \$0.6296. The asset swap spread is therefore $2 \times 0.6296 = 1.2592\%$ per annum.

Problem 22.13.

Show that the value of a coupon-bearing corporate bond is the sum of the values of its constituent zero-coupon bonds when the amount claimed in the event of default is the no-default value of the bond, but that this is not so when the claim amount is the face value of the bond plus accrued interest.

When the claim amount is the no-default value, the loss for a corporate bond arising from a default at time t is

$$v(t)(1 - \hat{R})B^*$$

where $v(t)$ is the discount factor for time t and B^* is the no-default value of the bond at time t. Suppose that the zero-coupon bonds comprising the corporate bond have no-default values at time t of Z_1, Z_2, ..., Z_n, respectively. The loss from the ith zero-coupon bond arising from a default at time t is

$$v(t)(1 - \hat{R})Z_i$$

The total loss from all the zero-coupon bonds is

$$v(t)(1 - \hat{R}) \sum_{i}^{n} Z_i = v(t)(1 - \hat{R})B^*$$

This shows that the loss arising from a default at time t is the same for the corporate bond as for the portfolio of its constituent zero-coupon bonds. It follows that the value of the corporate bond is the same as the value of its constituent zero-coupon bonds.

When the claim amount is the face value plus accrued interest, the loss for a corporate bond arising from a default at time t is

$$v(t)B^* - v(t)\hat{R}[L + a(t)]$$

where L is the face value and $a(t)$ is the accrued interest at time t. In general this is not the same as the loss from the sum of the losses on the constituent zero-coupon bonds.

Problem 22.14.

A four-year corporate bond provides a coupon of 4% per year payable semiannually and has a yield of 5% expressed with continuous compounding. The risk-free yield curve is flat at 3% with continuous compounding. Assume that defaults can take place at the end of each year (immediately before a coupon or principal payment and the recovery rate is 30%. Estimate the risk-neutral default probability on the assumption that it is the same each year.

Define Q as the risk-free rate. The calculations are as follows

Time (yrs)	Def. Prob.	Recovery Amount ($)	Risk-free Value ($)	Loss Given Default ($)	Discount Factor	PV of Expected Loss ($)
1.0	Q	30	104.78	74.78	0.9704	$72.57Q$
2.0	Q	30	103.88	73.88	0.9418	$69.58Q$
3.0	Q	30	102.96	72.96	0.9139	$66.68Q$
4.0	Q	30	102.00	72.00	0.8869	$63.86Q$
Total						$272.69Q$

The bond pays a coupon of 2 every six months and has a continuously compounded yield of 5% per year. Its market price is 96.19. The risk-free value of the bond is obtained by discounting the promised cash flows at 3%. It is 103.66. The total loss from defaults should therefore be equated to $103.66 - 96.19 = 7.46$. The value of Q implied by the bond price is therefore given by $272.69Q = 7.46$. or $Q = 0.0274$. The implied probability of default is 2.74% per year.

Problem 22.15.

A company has issued 3- and 5-year bonds with a coupon of 4% per annum payable annually. The yields on the bonds (expressed with continuous compounding) are 4.5%

and 4.75%, respectively. Risk-free rates are 3.5% with continuous compounding for all maturities. The recovery rate is 40%. Defaults can take place half way through each year. The risk-neutral default rates per year are Q_1 for years 1 to 3 and Q_2 for years 4 and 5. Estimate Q_1 and Q_2.

The table for the first bond is

Time (yrs)	Def. Prob.	Recovery Amount ($)	Risk-free Value ($)	Loss Given Default ($)	Discount Factor	PV of Expected Loss ($)
0.5	Q_1	40	103.01	63.01	0.9827	$61.92Q_1$
1.5	Q_1	40	102.61	62.61	0.9489	$59.41Q_1$
2.5	Q_1	40	102.20	62.20	0.9162	$56.98Q_1$
Total						$178.31Q_1$

The market price of the bond is 98.35 and the risk-free value is 101.23. It follows that Q_1 is given by

$$178.31Q_1 = 101.23 - 98.35$$

so that $Q_1 = 0.0161$.

The table for the second bond is

Time (yrs)	Def. Prob.	Recovery Amount ($)	Risk-free Value ($)	Loss Given Default ($)	Discount Factor	PV of Expected Loss ($)
0.5	Q_1	40	103.77	63.77	0.9827	$62.67Q_1$
1.5	Q_1	40	103.40	63.40	0.9489	$60.16Q_1$
2.5	Q_1	40	103.01	63.01	0.9162	$57.73Q_1$
3.5	Q_2	40	102.61	62.61	0.8847	$55.39Q_2$
4.5	Q_2	40	102.20	62.20	0.8543	$53.13Q_2$
Total						$180.56Q_1 + 108.53Q_2$

The market price of the bond is 96.24 is and the risk-free value is 101.97. It follows that

$$180.56Q_1 + 108.53Q_2 = 101.97 - 96.24$$

From which we get $Q_2 = 0.0260$ The bond prices therefore imply a probability of default of 1.61% per year for the first three years and 2.60% for the next two years.

Problem 22.16.

Suppose that a financial institution has entered into a swap dependent on the sterling interest rate with counterparty X and an exactly offsetting swap with counterparty Y. Which of the following statements are true and which are false.

(a) The total present value of the cost of defaults is the sum of the present value of the cost of defaults on the contract with X plus the present value of the cost of defaults on the contract with Y.
(b) The expected exposure in one year on both contracts is the sum of the expected exposure on the contract with X and the expected exposure on the contract with Y.
(c) The 95% upper confidence limit for the exposure in one year on both contracts is the sum of the 95% upper confidence limit for the exposure in one year on the contract with X and the 95% upper confidence limit for the exposure in one year on the contract with Y.
Explain your answers.

The statements in (a) and (b) are true. The statement in (c) is not. Suppose that v_X and v_Y are the exposures to X and Y. The expected value of $v_X + v_Y$ is the expected value of v_X plus the expected value of v_Y. The same is not true of 95% confidence limits.

Problem 22.17.

A company enters into a one-year forward contract to sell \$100 for AUD150. The contract is initially at the money. In other words, the forward exchange rate is 1.50. The one-year dollar risk-free rate of interest is 5% per annum. The one-year dollar rate of interest at which the counterparty can borrow is 6% per annum. The exchange rate volatility is 12% per annum. Estimate the present value of the cost of defaults on the contract? Assume that defaults are recognized only at the end of the life of the contract.

The cost of defaults is uv where u is percentage loss from defaults during the life of the contract and v is the value of an option that pays off $\max(150S_T - 100, 0)$ in one year and S_T is the value in dollars of one AUD. The value of u is

$$u = 1 - e^{-(0.06-0.05)\times 1} = 0.009950$$

The variable v is 150 times a call option to buy one AUD for 0.6667. The formula for the call option in terms of forward prices is

$$[FN(d_1) - KN(d_2)]e^{-rT}$$

where

$$d_1 = \frac{\log(F/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

In this case $F = 0.6667$, $K = 0.6667$, $\sigma = 0.12$, $T = 1$, and $r = 0.05$ so that $d_1 = 0.06$,$d_2 = -0.06$ and the value of the call option is 0.0303. It follows that $v = 150 \times 0.0303 = 4.545$ so that the cost of defaults is

$$4.545 \times 0.009950 = 0.04522$$

Problem 22.18.

Suppose that in Problem 22.17, the six-month forward rate is also 1.50 and the six-month dollar risk-free interest rate is 5% per annum. Suppose further that the six-month dollar rate of interest at which the counterparty can borrow is 5.5% per annum. Estimate the present value of the cost of defaults assuming that defaults can occur either at the six-month point or at the one-year point? (If a default occurs at the six-month point, the company's potential loss is the market value of the contract.)

In this case the costs of defaults is $u_1v_1 + u_2v_2$ where

$$u_1 = 1 - e^{-(0.055-0.05)\times 0.5} = 0.002497$$

$$u_2 = e^{-(0.055-0.05)\times 0.5} - e^{-(0.06-0.05)\times 1} = 0.007453$$

v_1 is the value of an option that pays off $\max(150S_T - 100,\ 0)$ in six months and v_2 is the value of a option that pays off $\max(150S_T - 100,\ 0)$ in one year. The calculations in Problem 22.17 shows that v_2 is 4.545. Similarly $v_1 = 3.300$ so that the cost of defaults is

$$0.002497 \times 3.300 + 0.007453 \times 4.545 = 0.04211$$

Problem 22.19.

"A long forward contract subject to credit risk is a combination of a short position in a no-default put and a long position in a call subject to credit risk." Explain this statement.

Assume that defaults happen only at the end of the life of the forward contract. In a default-free world the forward contract is the combination of a long European call and a short European put where the strike price of the options equals the delivery price and the maturity of the options equals the maturity of the forward contract. If the no-default value of the contract is positive at maturity, the call has a positive value and the put is worth zero. The impact of defaults on the forward contract is the same as that on the call. If the no-default value of the contract is negative at maturity, the call has a zero value and the put has a positive value. In this case defaults have no effect. Again the impact of defaults on the forward contract is the same as that on the call. It follows that the contract has a value equal to a long position in a call that is subject to default risk and short position in a default-free put.

Problem 22.20.

Explain why the credit exposure on a matched pair of forward contracts resembles a straddle.

Suppose that the forward contract provides a payoff at time T. With our usual notation, the value of a long forward contract is $S_T - Ke^{-rT}$. The credit exposure on a long forward contract is therefore $\max(S_T - Ke^{-rT},\ 0)$; that is, it is a call on the asset price with strike price Ke^{-rT}. Similarly the credit exposure on a short forward contract is $\max(Ke^{-rT} - S_T,\ 0)$; that is, it is a put on the asset price with strike price Ke^{-rT}. The total credit exposure is, therefore, a straddle with strike price Ke^{-rT}.

Problem 22.21.

Explain why the impact of credit risk on a matched pair of interest rate swaps tends to be less than that on a matched pair of currency swaps.

The credit risk on a matched pair of interest rate swaps is $|B_{\text{fixed}} - B_{\text{floating}}|$. As maturity is approached all bond prices tend to par and this tends to zero. The credit risk on a matched pair of currency swaps is $|SB_{\text{foreign}} - B_{\text{fixed}}|$ where S is the exchange rate. The expected value of this tends to increase as the swap maturity is approached because of the uncertainty in S.

Problem 22.22.

"When a bank is negotiating currency swaps, it should try to ensure that it is receiving the lower interest rate currency from a company with a low credit risk." Explain.

As time passes there is a tendency for the currency which has the lower interest rate to strengthen. This means that a swap where we are receiving this currency will tend to move in the money (i.e., have a positive value). Similarly a swap where we are paying the currency will tend to move out of the money (i.e., have a negative value). From this it follows that our expected exposure on the swap where we are receiving the low-interest currency is much greater than our expected exposure on the swap where we are receiving the high-interest currency. We should therefore look for counterparties with a low credit risk on the side of the swap where we are receiving the low-interest currency. On the other side of the swap we are far less concerned about the creditworthiness of the counterparty.

Problem 22.23.

Does put–call parity hold when there is default risk? Explain your answer.

No, put–call parity does not hold when there is default risk. Suppose c^* and p^* are the no-default prices of a European call and put with strike price K and maturity T on a non-dividend-paying stock whose price is S, and that c and p are the corresponding values when there is default risk. The text shows that when we make the independence assumption (that is, we assume that the variables determining the no-default value of the option are independent of the variables determining default probabilities and recovery rates), $c = c^* e^{-[y(T)-y^*(T)]T}$ and $p = p^* e^{-[y(T)-y^*(T)]T}$. The relationship

$$c^* + Ke^{-y^*(T)T} = p^* + S$$

which holds in a no-default world therefore becomes

$$c + Ke^{-y(T)T} = p + Se^{-[y(T)-y^*(T)]T}$$

when there is default risk. This is not the same a regular put–call parity. What is more, the relationship depends on the independence assumption and cannot be deduced from the same sort of simple no-arbitrage arguments that we used in Chapter 9 for the put–call parity relationship in a no-default world.

Problem 22.24.

Suppose that in an asset swap B is the market price of the bond per dollar of principal, B^* is the default-free value of the bond per dollar of principal, and V is the present value of the asset swap spread per dollar of principal. Show that $V = B^* - B$.

We can assume that the principal is paid and received at the end of the life of the swap without changing the swap's value. If the spread were zero the present value of the floating payments per dollar of principal would be 1. The payment of LIBOR plus the spread therefore has a present value of $1 + V$. The payment of the bond cash flows has a present value per dollar of principal of B^*. The inital payment required from the payer of the bond cash flows per dollar of principal is $1 - B$. (This may be negative; an initial amount of $B - 1$ is then paid by the payer of the floating rate). Because the asset swap is initially worth zero we have

$$1 + V = B^* + 1 - B$$

so that

$$V = B^* - B$$

Problem 22.25.

Show that under Merton's model in Section 22.6 the credit spread on a T-year zero-coupon bond is $-\ln[N(d_2) + N(-d_1)/L]/T$ where $L = De^{-rT}/V_0$.

The value of the debt in Merton's model is $V_0 - E_0$ or

$$De^{-rT}N(d_2) - V_0N(d_1) + V_0 = De^{-rT}N(d_2) + V_0N(-d_1)$$

If the credit spread is s this should equal $De^{-(r+s)T}$ so that

$$De^{-(r+s)T} = De^{-rT}N(d_2) + V_0N(-d_1)$$

Substituting $De^{-rT} = LV_0$

$$LV_0e^{-sT} = LV_0N(d_2) + V_0N(-d_1)$$

or

$$Le^{-sT} = LN(d_2) + N(-d_1)$$

so that

$$s = -\ln[N(d_2) + N(-d_1)/L]/T$$

Problem 22.26.

Suppose that the spread between the yield on a 3-year zero-coupon riskless bond and a 3-year zero-coupon bond issued by a corporation is 1%. By how much does Black–Scholes overstate the value of a 3-year European option sold by the corporation.

When the default risk of the seller of the option is taken into account the option value is the Black–Scholes price multiplied by $e^{-0.01\times3} = 0.9704$. Black–Scholes overprices the option by about 3%.

Problem 22.27

Give an example of a) right-way risk and b) wrong-way risk.

(a) Right way risk describes the situation when a default by the counterparty is most likely to occur when the contract has a positive value to the counterparty. A example of right way risk would be when a counterparty's future depends on the price of a commodity and it enters into a contract to partially hedging that exposure.

(b) Wrong way risk describes the situation when a default by the counterparty is most likely to occur when the contract has a negative value to the counterparty. A example of right way risk would be when a counterparty is a speculator and the contract has the same exposure as the rest of the counterparty's portfolio.

ASSIGNMENT QUESTIONS

Problem 22.28.

Suppose a three-year corporate bond provides a coupon of 7% per year payable semi-annually and has a yield of 5% (expressed with semiannual compounding). The yields for all maturities on risk-free bonds is 4% per annum (expressed with semiannual compuinding). Assume that defaults can take place every six months (immediately before a coupon payment) and the recovery rate is 45%. Estimate the default probabilities assuming a) the unconditional default probabilities are the same on each possible default date and b) assuming that the default probabilities conditional on no earlier default are the same on each possible default date.

(a) The market price of the bond is 105.51. The risk-free price is 108.40. The expected cost of defaults is therefore 2.89. The following table calculates the cost of defaults as 348.20Q where Q is the unconditional probability of default each year. This means that the probability of default per year is 2.89/348.20 or 0.00831.

Time (yrs)	Def. Prob.	Recovery Amount ($)	Risk-free Value ($)	Loss Given Default ($)	Discount Factor	PV of Expected Loss ($)
0.5	Q	45	110.57	65.57	0.9804	$64.28Q$
1.0	Q	45	109.21	64.21	0.9612	$61.73Q$
1.5	Q	45	107.83	62.83	0.9423	$59.20Q$
2.0	Q	45	106.41	61.41	0.9238	$56.74Q$
2.5	Q	45	104.97	59.97	0.9057	$54.32Q$
3.0	Q	45	103.50	58.50	0.8880	$51.95Q$
Total						$348.20Q$

(b) Suppose that Q^* is the default probability conditional on no earlier default. The unconditional default probabilities in 0.5, 1.0, 1.5, 2.0, 2.5, 3.0 years are Q^*, $Q^*(1-Q^*)$, $Q^*(1-Q^*)^2$, $Q^*(1-Q^*)^3$, $Q^*(1-Q^*)^4$, $Q^*(1-Q^*)^5$. We must therefore find the value of Q^* that solves

$$64.28Q^* + 61.73Q^*(1 - Q^*) + 59.20 * Q^*(1 - Q^*)^2 + 56.74Q^*(1 - Q^*)^3$$

$$+54.32Q^*(1 - Q^*)^4 + 51.95Q^*(1 - Q^*)^5 = 2.89$$

Using Solver in Excel we find that $Q^* = 0.00848$.

Problem 22.29.

A company has one- and two-year bonds outstanding, each providing a coupon of 8% per year payable annually. The yields on the bonds (expressed with continuous compounding are 6.0% and 6.6%, respectively. Risk-free rates are 4.5% for all maturities. The recovery rate is 35%. Defaults can take place half way through each year. Estimate the risk-neutral default rate each year.

Consider the first bond. Its market price is $108e^{-0.06\times 1} = 101.71$. Its default-free price is $108e^{-0.045\times 1} = 103.25$. The present value of the loss from defaults is therefore 1.54. In this case losses can take place at only one time, half way through the year. Suppose that the probability of default at this time is Q_1. The default-free value of the bond is $108e^{-0.045\times 0.5} = 105.60$. The loss in the event of a default is $105.60 - 35 = 70.60$. The present value of the expected loss is $70.60e^{-0.045\times 0.5}Q_1 = 69.03Q_1$. It follows that

$$69.03Q_1 = 1.54$$

so that $Q_1 = 0.0223$.

Now consider the second bond. It market price is 103.32 and its default-free value is 106.35. The present value of the loss from defaults is therefore 3.03. At time 0.5 the default free value of the bond is 108.77. The loss in the event of a default is therefore 73.77. The present value of the loss from defaults at this time is $72.13Q_1$ or 1.61. This means that the present value of the loss from defaults at the 1.5 year point is $3.03 - 1.61$ or 1.42. The default-free value of the bond at the 1.5 year point is 105.60. The loss in the event of a default is 70.60. The present value of the expected loss is $65.99Q_2$ where Q_2 is the probability of default in the second year. It follows that

$$65.99Q_2 = 1.42$$

so that $Q_2 = 0.0216$.

The probabilities of default in years one and two are therefore 2.23% and 2.16%.

Problem 22.30.

Explain carefully the distinction between real-world and risk-neutral default probabilities. Which is higher? A bank enters into a credit derivative where it agrees to pay \$100 at the end of one year if a certain company's credit rating falls from A to Baa or lower during the year. The one-year risk-free rate is 5%. Using Table 22.6, estimate a value for the derivative. What assumptions are you making? Do they tend to overstate or understate the value of the derivative.

Real world default probabilities are the true probabilities of defaults. They can be estimated from historical data. Risk-neutral default probabilities are the probabilities of defaults in a world where all market participants are risk neutral. They can be estimated from bond prices. Risk-neutral default probabilities are higher. This means that returns in the risk-neutral world are lower. From Table 22.6 the probability of a company moving from A to Baa or lower in one year is 5.73%. An estimate of the value of the derivative is therefore $0.0573 \times 100 \times e^{-0.05\times 1} = 5.45$. The approximation in this is that we are using the real-world probability of a downgrade. To value the derivative correctly we should use the risk-neutral probability of a downgrade. Since the risk-neutral probability of a default is higher than the real-world probability, it seems likely that the same is true of a downgrade. This means that 5.45 is likely to be too low as an estimate of the value of the derivative.

Problem 22.31.

The value of a company's equity is \$4 million and the volatility of its equity is 60%. The debt that will have to be repaid in two years is \$15 million. The risk-free interest rate is 6% per annum. Use Merton's model to estimate the expected loss from default, the probability of default, and the recovery rate in the event of default. Explain why Merton's model gives a high recovery rate. (Hint The Solver function in Excel can be used for this question.)

In this case $E_0 = 4$, $\sigma_E = 0.60$, $D = 15$, $r = 0.06$. Setting up the data in Excel, we can solve equations (22.3) and (22.4) by using the approach in footnote 12. The solution to the equations proves to be $V_0 = 17.084$ and $\sigma_V = 0.1576$. The probability of default is $N(-d_2)$ or 15.61%. The market value of the debt is $17.084 - 4 = 13.084$. The present value of the promised payment on the debt is $15e^{-0.06\times 2} = 13.304$. The expected loss on the debt is, therefore, $(13.304 - 13.084)/13.304$ or 1.65% of its no-default value. The expected recovery rate in the event of default is therefore $(15.61 - 1.65)/15.61$ or about 89%. The reason the recovery rate is so high is as follows. There is a default if the value of the assets moves from 17.08 to below 15. A value for the assets significantly below 15 is unlikely. Conditional on a default, the expected value of the assets is, therefore, not a huge amount below 15. In practice it is likely that companies manage to delay defaults until asset values are well below the face value of the debt.

Problem 22.32.

Suppose that a bank has a total of \$10 million of exposures of a certain type. The one-year probability of default averages 1% and the recovery rate averages 40%. The copula correlation parameter is 0.2. Estimate the 99.5% one-year credit VaR.

From equation (22.11) the 99.5% worst case probability of default is

$$N\left(\frac{N^{-1}(0.01) + \sqrt{0.2}N^{-1}(0.995)}{\sqrt{0.8}}\right) = 0.0946$$

This gives the 99.5% credit VaR as $10 \times (1 - 0.4) \times 0.0946 = 0.568$ millions of dollars or \$568,000.

CHAPTER 23
Credit Derivatives

Notes for the Instructor

Chapter 23 has been significantly revised for the seventh edition. Part of the chapter is now devoted to asset-backed securities and a discussion of the credit crunch of 2007. This comes before the material on CDOs. There is now more material (with numerical examples) on the use of the one-factor Gaussian copula model to value CDOs.

Credit derivatives are an exciting and relatively recent innovation. They have the potential to allow organizations to manage their credit risks in much the same way that they manage market risks. Students generally enjoy the material in Chapter 23. Indeed many quickly decide that they would love to work in the credit derivatives area because of the pace of innovation.

This chapter follows on from Chapter 22. It starts by discussing in some detail how credit default swaps work and how they are valued. It then moves on to binary CDSs, CDS forwards and options, total return swaps, basket credit default swaps, asset-backed securities, and collateralized debt obligations. The section on convertible bonds has been moved to Chapter 26.

Problem 23.27, 23.28, and 23.29 can be discussed in class. Others work well as assignment questions. Problems 23.30 and 23.31 are more challenging.

QUESTIONS AND PROBLEMS

Problem 23.1.

Explain the difference between a regular credit default swap and a binary credit default swap.

Both provide insurance against a particular company defaulting during a period of time. In a credit default swap the payoff is the notional principal amount multiplied by one minus the recovery rate. In a binary swap the payoff is the notional principal.

Problem 23.2.

A credit default swap requires a semiannual payment at the rate of 60 basis points per year. The principal is \$300 million and the credit default swap is settled in cash. A default occurs after four years and two months, and the calculation agent estimates that the price of the cheapest deliverable bond is 40% of its face value shortly after the default. List the cash flows and their timing for the seller of the credit default swap.

The seller receives

$$300,000,000 \times 0.0060 \times 0.5 = \$900,000$$

at times 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, and 4.0 years. The seller also receives a final accrual payment of about \$300,000 (= $\$300,000,000 \times 0.060 \times 2/12$) at the time of the default (4 years and two months). The seller pays

$$300,000,000 \times 0.6 = \$180,000,000$$

at the time of the default.

Problem 23.3.

Explain the two ways a credit default swap can be settled.

Sometimes there is physical settlement and sometimes there is cash settlement. In the event of a default when there is physical settlement the buyer of protection sells bonds issued by the reference entity for their face value. Bonds with a total face value equal to the notional principal can be sold. In the event of a default when there is cash settlement a calculation agent estimates the value of the cheapest-to-deliver bonds issued by the reference entity a specified number of days after the default event. The cash payoff is then based on the excess of the face value of these bonds over the estimated value.

Problem 23.4.

Explain how a CDO and a Synthetic CDO are created.

A CDO is created from a bond portfolio. The returns from the bond portfolio flow to a number of tranches (i.e., different categories of investors). The tranches differ as far as the credit risk they assume. The first tranche might have an investment in 5% of the bond portfolio and be responsible for the first 5% of losses. The next tranche might have an investment in 10% of the portfolio and be responsible for the next 10% of the losses, and so on. In a synthetic CDO there is no bond portfolio. Instead a portfolio of credit default swaps is sold and the resulting credit risks are allocated to tranches in a similar way to that just described.

Problem 23.5.

Explain what a first-to-default credit default swap is. Does its value increase or decrease as the default correlation between the companies in the basket increases? Explain.

In a first-to-default basket CDS there are a number of reference entities. When the first one defaults there is a payoff (calculated in the usual way for a CDS) and basket CDS terminates. The value of a first-to-default basket CDS decreases as the correlation between the reference entities in the basket increases. This is because the probability of a default is high when the correlation is zero and decreases as the correlation increases. In the limit when the correlation is one there is in effect only one company and the probability of a default is quite low.

Problem 23.6.

Explain the difference between risk-neutral and real-world default probabilities.

Risk-neutral default probabilities are backed out from credit default swaps or bond prices. Real-world default probabilities are calculated from historical data.

Problem 23.7.

Explain why a total return swap can be useful as a financing tool.

Suppose a company wants to buy some assets. If a total return swap is used, a financial institution buys the assets and enters into a swap with the company where it pays the company the return on the assets and receives from the company LIBOR plus a spread. The financial institution has less risk than it would have if it lent the company money and used the assets as collateral. This is because, in the event of a default by the company it owns the assets.

Problem 23.8.

Suppose that the risk-free zero curve is flat at 7% per annum with continuous compounding and that defaults can occur half way through each year in a new five-year credit default swap. Suppose that the recovery rate is 30% and the default probabilities each year conditional on no earlier default is 3% Estimate the credit default swap spread. Assume payments are made annually.

The table corresponding to Tables 23.1, giving unconditional default probabilities, is

Time (years)	Default Probability	Survival Probability
1	0.0300	0.9700
2	0.0291	0.9409
3	0.0282	0.9127
4	0.0274	0.8853
5	0.0266	0.8587

The table corresponding to Table 23.2, giving the present value of the expected regular payments (payment rate is s per year), is

Time (years)	Probability of Survival	Expected Payment	Discount Factor	PV of Expected Payment
1	0.9700	$0.9700s$	0.9324	$0.9044s$
2	0.9409	$0.9409s$	0.8694	$0.8180s$
3	0.9127	$0.9127s$	0.8106	$0.7398s$
4	0.8853	$0.8853s$	0.7558	$0.6691s$
5	0.8587	$0.8587s$	0.7047	$0.6051s$
Total				$3.7364s$

The table corresponding to Table 23.3, giving the present value of the expected payoffs (notional principal =$1), is

Time (years)	Probability of Default	Recovery Rate	Expected Payoff	Discount Factor	PV of Expected Payoff
0.5	0.0300	0.3	0.0210	0.9656	0.0203
1.5	0.0291	0.3	0.0204	0.9003	0.0183
2.5	0.0282	0.3	0.0198	0.8395	0.0166
3.5	0.0274	0.3	0.0192	0.7827	0.0150
4.5	0.0266	0.3	0.0186	0.7298	0.0136
Total					0.0838

The table corresponding to Table 23.4, giving the present value of accrual payments, is

Time (years)	Probability of Default	Expected Accrual Payment	Discount Factor	PV of Expected Accrual Payment
0.5	0.0300	$0.0150s$	0.9656	$0.0145s$
1.5	0.0291	$0.0146s$	0.9003	$0.0131s$
2.5	0.0282	$0.0141s$	0.8395	$0.0118s$
3.5	0.0274	$0.0137s$	0.7827	$0.0107s$
4.5	0.0266	$0.0133s$	0.7298	$0.0097s$
Total				$0.0598s$

The credit default swap spread s is given by:

$$3.7364s + 0.0598s = 0.0838$$

It is 0.0221 or 221 basis points.

Problem 23.9.

What is the value of the swap in Problem 23.8 per dollar of notional principal to the protection buyer if the credit default swap spread is 150 basis points?

If the credit default swap spread is 150 basis points, the value of the swap to the buyer of protection is:

$$0.0838 - (3.7364 + 0.0598) \times 0.0150 = 0.0269$$

per dollar of notional principal.

Problem 23.10.

What is the credit default swap spread in Problem 23.8 if it is a binary CDS?

If the swap is a binary CDS, the present value of expected payoffs is calculated as follows

Time (years)	Probability of Default	Expected Payoff	Discount Factor	PV of Expected Payoff
0.5	0.0300	0.0300	0.9656	0.0290
1.5	0.0291	0.0291	0.9003	0.0262
2.5	0.0282	0.0282	0.8395	0.0237
3.5	0.0274	0.0274	0.7827	0.0214
4.5	0.0266	0.0266	0.7298	0.0194
Total				0.1197

The credit default swap spread s is given by:

$$3.7364s + 0.0598s = 0.1197$$

It is 0.0315 or 315 basis points.

Problem 23.11.

How does a five-year nth-to-default credit default swap work? Consider a basket of 100 reference entities where each reference entity has a probability of defaulting in each year of 1%. As the default correlation between the reference entities increases what would you expect to happen to the value of the swap when a) $n = 1$ and b) $n = 25$. Explain your answer.

A five-year nth to default credit default swap works in the same way as a regular credit default swap except that there is a basket of companies. The payoff occurs when the nth default from the companies in the basket occurs. After the nth default has occurred the swap ceases to exist. When $n = 1$ (so that the swap is a "first to default") an increase in the default correlation lowers the value of the swap. When the default correlation is zero there are 100 independent events that can lead to a payoff. As the correlation increases the probability of a payoff decreases. In the limit when the correlation is perfect there is in effect only one company and therefore only one event that can lead to a payoff.

When $n = 25$ (so that the swap is a 25th to default) an increase in the default correlation increases the value of the swap. When the default correlation is zero there is virtually no chance that there will be 25 defaults and the value of the swap is very close to zero. As the correlation increases the probability of multiple defaults increases. In the limit when the correlation is perfect there is in effect only one company and the value of a 25th-to-default credit default swap is the same as the value of a first-to-default swap.

Problem 23.12.

What is the formula relating the payoff on a CDS to the notional principal and the recovery rate?

The payoff is $L(1-R)$ where L is the notional principal and R is the recovery rate.

Problem 23.13.

Show that the spread for a new plain vanilla CDS should be $(1-R)$ times the spread for a similar new binary CDS where R is the recovery rate.

The payoff from a plain vanilla CDS is $1-R$ times the payoff from a binary CDS with the same principal. The payoff always occurs at the same time on the two instruments. It follows that the regular payments on a new plain vanilla CDS must be $1-R$ times the payments on a new binary CDS. Otherwise there would be an arbitrage opportunity.

Problem 23.14.

Verify that if the CDS spread for the example in Tables 23.1 to 21.4 is 100 basis points and the probability of default in a year (conditional on no earlier default) must be 1.61%. How does the probability of default change when the recovery rate is 20% instead of 40%? Verify that your answer is consistent with the implied probability of default being approximately proportional to $1/(1-R)$ where R is the recovery rate.

The 1.61% implied default probability can be calculated by setting up a worksheet in Excel and using Solver. To verify that 1.61% is correct we note that, with a conditional default probability of 1.61%, the unconditional probabilities are:

Time (years)	Default Probability	Survival Probability
1	0.0161	0.9839
2	0.0158	0.9681
3	0.0156	0.9525
4	0.0153	0.9371
5	0.0151	0.9221

The present value of the regular payments becomes $4.1170s$, the present value of the expected payoffs becomes 0.0415, and the present value of the expected accrual payments becomes $0.0346s$. When $s = 0.01$ the present value of the expected payments equals the present value of the expected payoffs.

When the recovery rate is 20% the implied default probability (calculated using Solver) is 1.21% per year. Note that 1.21/1.61 is approximately equal to $(1-0.4)/(1-0.2)$ showing that the implied default probability is approximately proportional to $1/(1-R)$.

In passing we note that if the CDS spread is used to imply an unconditional default probability (assumed to be the same each year) then this implied unconditional default probability is exactly proportional to $1/(1-R)$. When we use the CDS spread to imply a conditional default probability (assumed to be the same each year) it is only approximately proportional to $1/(1-R)$.

Problem 23.15.

A company enters into a total return swap where it receives the return on a corporate bond paying a coupon of 5% and pays LIBOR. Explain the difference between this and a regular swap where 5% is exchanged for LIBOR.

In the case of a total return swap a company receives (pays) the increase (decrease) in the value of the bond. In the regular swap this does not happen.

Problem 23.16.

Explain how forward contracts and options on credit default swaps are structured.

When a company enters into a long (short) forward contract it is obligated to buy (sell) the protection given by a specified credit default swap with a specified spread at a specified future time. When a company buys a call (put) option contract it has the option to buy (sell) the protection given by a specified credit default swap with a specified spread at a specified future time. Both contracts are normally structured so that they cease to exist if a default occurs during the life of the contract.

Problem 23.17.

"The position of a buyer of a credit default swap is similar to the position of someone who is long a risk-free bond and short a corporate bond." Explain this statement.

A credit default swap insures a corporate bond issued by the reference entity against default. Its approximate effect is to convert the corporate bond into a risk-free bond. The buyer of a credit default swap has therefore chosen to exchange a corporate bond for a risk-free bond. This means that the buyer is long a risk-free bond and short a similar corporate bond.

Problem 23.18.

Why is there a potential asymmetric information problem in credit default swaps?

Payoffs from credit default swaps depend on whether a particular company defaults. Arguably some market participants have more information about this that other market participants. (See Business Snapshot 23.2.)

Problem 23.19.

Does valuing a CDS using real-world default probabilities rather than risk-neutral default probabilities overstate or understate its value? Explain your answer.

Real world default probabilities are less than risk-neutral default probabilities. It follows that the use of actuarial default probabilities will tend to understate the value of a CDS.

Problem 23.20.

What is the difference between a total return swap and an asset swap?

In an asset swap the bond's promised payments are swapped for LIBOR plus a spread. In a total return swap the bond's actual payments are swapped for LIBOR plus a spread.

Problem 23.21.

Suppose that in a one-factor Gaussian copula model the five-year probability of default for each of 125 names is 3% and the pairwise copula correlation is 0.2. Calculate, for factor values of –2, –1, 0, 1, and 2, a) the default probability conditional on the factor value and b) the probability of more than 10 defaults conditional on the factor value.

Using equation (23.2) the probability of default conditional on a factor value of M is

$$N\left(\frac{N^{-1}(0.03)-\sqrt{0.2}M}{\sqrt{1-0.2}}\right)$$

For M equal to –2, –1, 0, 1, and 2 the probabilities of default are 0.135, 0.054, 0.018, 0.005, and 0.001 respectively. To six decimal places the probability of more that 10 defaults for these values of M can be calculated using the BINOMDIST function in Excel. They are 0.959284, 0.79851, 0.000016, 0, and 0, respectively.

Problem 23.22.

Explain the difference between base correlation and compound correlation

Compound correlation for a tranche is the correlation which when substituted into the one-factor Gaussian copula model produces the market quote for the tranche. Base correlation is the correlation which is consistent with the one-factor Gaussian copula and market quotes for the 0 to X% tranche where X% is a detachment point. It ensures that the expected loss on the 0 to X% tranche equals the sum of the expected losses on the underlying traded tranches.

Problem 23.23.

In the ABS CDO structure in Figure 23.4, suppose that there is a 12% loss on each portfolio. What is the percentage loss experienced by each of the six tranches shown.

The ABS equity tranche is wiped out. There are no losses to the senior ABS tranche. The ABS mezzanine tranche loses 7/20 = 35% of the principal.

Total losses on the ABS CDO are 35%. The ABS CDO equity and mezzanine tranches are wiped out. The ABS CDO senior tranche loses 10/75 = 13.3% of the principal.

Problem 23.24.

In Example 23.2, what is the tranche spread for the 9% to 12% tranche?

In this case $a_L = 0.09$ and $a_H = 0.12$. Proceeding similarly in Example 23.2 the tranche spread is calculated as 30 basis points.

ASSIGNMENT QUESTIONS

Problem 23.25.

Suppose that the risk-free zero curve is flat at 6% per annum with continuous compounding and that defaults can occur at times 0.25 years, 0.75 years, 1.25 years, and 1.75 years in a two-year plain vanilla credit default swap with semiannual payments. Suppose that the recovery rate is 20% and the unconditional probabilities of default (as seen at time zero) are 1% at times 0.25 years and 0.75 years, and 1.5% at times 1.25 years and 1.75 years. What is the credit default swap spread? What would the credit default spread be if the instrument were a binary credit default swap?

The table corresponding to Table 23.2, giving the present value of the expected regular payments (payment rate is s per year), is

Time (years)	Probability of Survival	Expected Payment	Discount Factor	PV of Expected Payment
0.5	0.990	$0.4950s$	0.9704	$0.4804s$
1.0	0.980	$0.4900s$	0.9418	$0.4615s$
1.5	0.965	$0.4825s$	0.9139	$0.4410s$
2.0	0.950	$0.4750s$	0.8869	$0.4213s$
Total				$1.8041s$

The table corresponding to Table 23.3, giving the present value of the expected payoffs (notional principal =\$1), is

Time (years)	Probability of Default	Recovery Rate	Expected Payoff	Discount Factor	PV of Expected Payoff
0.25	0.010	0.2	0.008	0.9851	0.0079
0.75	0.010	0.2	0.008	0.9560	0.0076
1.25	0.015	0.2	0.008	0.9277	0.0111
1.75	0.015	0.2	0.008	0.9003	0.0108
Total					0.0375

The table corresponding to Table 23.4, giving the present value of accrual payments, is

Time (years)	Probability of Default	Expected Accrual Payment	Discount Factor	PV of Expected Accrual Payment
0.25	0.010	$0.0025s$	0.9851	$0.0025s$
0.75	0.010	$0.0025s$	0.9560	$0.0024s$
1.25	0.015	$0.00375s$	0.9277	$0.0035s$
1.75	0.015	$0.00375s$	0.9003	$0.0034s$
Total				$0.0117s$

The credit default swap spread s is given by:

$$1.804s + 0.0117s = 0.0375$$

It is 0.0206 or 206 basis points. For a binary credit default swap we set the recovery rate equal to zero in the second table to get the present value of expected payoffs equal to 0.0468 so that

$$1.804s + 0.0117s = 0.0468$$

and the spread is 0.0258 or 258 basis points.

Problem 23.26.

Assume that the default probability for a company in a year, conditional on no earlier defaults is λ and the recovery rate is R. The risk-free interest rate is 5% per annum. Default always occur half way through a year. The spread for a five-year plain vanilla CDS where payments are made annually is 120 basis points and the spread for a five-year binary CDS where payments are made annually is 160 basis points. Estimate R and λ.

The spread for a binary credit default swap is equal to the spread for a regular credit default swap divided by $1 - R$ where R is the recovery rate. This means that $1 - R$ equals 0.75 so that the recovery rate is 25%. To find λ we search for the conditional annual default rate that leads to the present value of payments being equal to the present value of payoffs. The answer is $\lambda = 0.0154$. The present value of payoffs (per dollar of principal) is then 0.0497. The present value of regular payments is 0.0495. The present value of accrual payments is 0.0002.

Problem 23.27.

Explain how you would expect the yields offered on the various tranches in a CDO to change when the correlation between the bonds in the portfolio increases.

As the correlation increases the yield on the equity tranche decreases and the yield on the senior tranches increases. To understand this consider what happens as the correlation increases from zero to one. Initially the equity tranche is mush more risky than the senior tranche. But as the correlation approaches one the companies become essentially the same. We are then in the position where either all companies default or no companies default and the tranches have similar risk.

Problem 23.28.

Suppose that

a. The yield on a five-year risk-free bond is 7%

b. The yield on a five-year corporate bond issued by company X is 9.5%

c. A five-year credit default swap providing insurance against company X defaulting costs 150 basis points per year.

What arbitrage opportunity is there in this situation? What arbitrage opportunity would there be if the credit default spread were 300 basis points instead of 150 basis points? Give two reasons why arbitrage opportunities such as those you identify are less than perfect.

When the credit default swap spread is 150 basis points, an arbitrageur can earn more than the risk-free rate by buying the corporate bond and buying protection. If the arbitrageur can finance trades at the risk-free rate (by shorting the riskless bond) it is possible to lock in an almost certain profit of 100 basis points. When the credit spread is 300 basis points the arbitrageur can short the corporate bond, sell protection and buy a risk free bond. This will lock in an almost certain profit of 50 basis points. The arbitrage is not perfect for a number of reasons:

(a) It assumes that both the corporate bond and the riskless bond are par yield bonds and that interest rates are constant. In practice the riskless bond may be worth more or less than par at the time of a default so that a credit default swap underprotects or overprotects the bond holder relative to the position he or she would be in with a riskless bond.

(b) There is uncertainty created by the cheapest-to-delivery bond option.

(c) To be a perfect hedge the credit default swap would have to give the buyer of protection the right to sell the bond for face value plus accrued interest, not just face value.

(d) The arbitrage opportunities assume that market participants can short corporate bonds and borrow at the risk-free rate.

(e) The definition of the credit event in the ISDA agreement is also occasionally a problem. It can occasionally happen that there is a "credit event" but promised payments on the bond are made.

Problem 23.29.

In the ABS CDO structure in Figure 23.4, suppose that there is a 20% loss on each portfolio. What is the percentage loss experienced by each of the six tranches shown.

The ABS equity tranche is wiped out. There are no losses to the senior ABS tranche. The ABS mezzanine tranche loses $15/20 = 75\%$ of the principal.

Total losses on the ABS CDO are 75%. The ABS CDO equity and mezzanine tranches are wiped out. The ABS CDO senior tranche loses $50/75 = 66.7\%$ of the principal.

Problem 23.30.

In Example 23.3, what is the spread for a) a first-to-default CDS and b) a second-to-default CDS?

(a) In This case the answer to Example 23.3 gets modified as follows. When $F = -1.0104$ the cumulative probabilities of one or more defaults in 1, 2, 3, 4, and 5 years are 0.3103, 0.5435, 0.6997, 0.8027, and 0.8703. The conditional probability that the first default occurs in years 1, 2, 3, 4, and 5 are 0.3103, 0.2332, 0.1562, 0.1030, and 0.0676, respectively. The present values of payoffs, regular payments, and accrual payments conditional on $F = -1.0104$ are 0.4784, $1.5900s$, and $0.3987s$. Similar calculations are carried out for the other factor values. The unconditional expected present values of payoffs, regular payments, and accrual payments are 0.2618, $2.9230s$, and $0.2182s$. The breakeven spread is therefore

$$0.2618/(2.9230 + 0.2182) = 0.0833$$

or 833 basis points. (b) In this case the answer to Example 23.3 gets modified as follows. When $F = -1.0104$ the cumulative probabilities of two or more defaults in 1, 2, 3, 4, and

5 years are 0.0493, 0.1711, 0.3159, 0.4551, and 0.5765. The conditional probability that the second default occurs in years 1, 2, 3, 4, and 5 are 0.0493, 0.1219, 0.1447, 0.1392, and 0.1214, respectively. The present values of payoffs, regular payments, and accrual payments conditional on $F = -1.0104$ are 0.3016, 3.0192s, and 0.2513s. Similar calculations are carried out for the other factor values. The unconditional expected present values of payoffs, regular payments, and accrual payments are 0.1277, 3.7364s, and 0.1064s. The breakeven spread is therefore

$$0.1277/(3.7364 + 0.1064) = 0.0332$$

or 332 basis points.

Problem 23.31.

In Example 23.2, what is the tranche spread for the 6% to 9% tranche?

In this case $a_L = 0.06$ and $a_H = 0.0.09$. Proceeding similarly in Example 23.2 the tranche spread is calculated as 64 basis points.

CHAPTER 24
Exotic Options

Notes for the Instructor

This chapter now contains material on the variance swaps and volatility swaps, which are becoming increasingly popular. Section 24.13 includes numerical examples to show how they are valued. The calculation of the VIX index is also explained in this section. Anotehr new feature of the chapter is that it distinguishes between fixed and floating lookback options.

This chapter describes a wide range of different exotic options and presents analytic and approximate analytic valuations when these are available. It also discusses static option replication. I do not emphasize the formulas in the chapter. I prefer to spend time talking about how the options work and explaining their properties. Sometimes I use DerivaGem to illustrate the properties of particular options. Problem 24.25, for example, uses DerivaGem to investigate the properties of up-and-out barrier call options.

Problem 24.25 can be a useful part of class discussion when binary options are discussed. Of the assignment questions, 24.27, 24.30, and 24.31 are relatively challenging.

QUESTIONS and PROBLEMS

Problem 24.1.

Explain the difference between a forward start option and a chooser option.

A forward start option is an option that is paid for now but will start at some time in the future. The strike price is usually equal to the price of the asset at the time the option starts. A chooser option is an option where, at some time in the future, the holder chooses whether the option is a call or a put.

Problem 24.2.

Describe the payoff from a portfolio consisting of a lookback call and a lookback put with the same maturity.

A Floating lookback call provides a payoff of $S_T - S_{\min}$. A lookback put provides a payoff of $S_{\max} - S_T$. A combination of a lookback call and a lookback put therefore provides a payoff of $S_{\max} - S_{\min}$.

Problem 24.3.

Consider a chooser option where the holder has the right to choose between a European call and a European put at any time during a two-year period. The maturity dates and strike prices for the calls and puts are the same regardless of when the choice is made. Is *it ever optimal to make the choice before the end of the two-year period? Explain your answer.*

No, it is never optimal to choose early. The resulting cash flows are the same regardless of when the choice is made. There is no point in the holder making a commitment earlier than necessary. This argument applies when the holder chooses between two American options providing the options cannot be exercised before the 2-year point. If the early exercise period starts as soon as the choice is made, the argument does not hold. For example, if the stock price fell to almost nothing in the first six months, the holder would choose a put option at this time and exercise it immediately.

Problem 24.4.

Suppose that c_1 and p_1 are the prices of a European average price call and a European average price put with strike price K and maturity T, c_2 and p_2 are the prices of a European average strike call and European average strike put with maturity T, and c_3 and p_3 are the prices of a regular European call and a regular European put with strike price K and maturity T. Show that

$$c_1 + c_2 - c_3 = p_1 + p_2 - p_3$$

The payoffs are as follows:
c_1 : $\max(\overline{S} - K, 0)$
c_2 : $\max(S_T - \overline{S}, 0)$
c_3 : $\max(S_T - K, 0)$
p_1 : $\max(K - \overline{S}, 0)$
p_2 : $\max(\overline{S} - S_T, 0)$
p_3 : $\max(K - S_T, 0)$
The payoff from $c_1 - p_1$ is always $\overline{S} - K$; The payoff from $c_2 - p_2$ is always $S_T - \overline{S}$; The payoff from $c_3 - p_3$ is always $S_T - K$; It follows that

$$c_1 - p_1 + c_2 - p_2 = c_3 - p_3$$

or

$$c_1 + c_2 - c_3 = p_1 + p_2 - p_3$$

Problem 24.5.

The text derives a decomposition of a particular type of chooser option into a call maturing at time T_2 and a put maturing at time T_1. Derive an alternative decomposition into a call maturing at time T_1 and a put maturing at time T_2.

Substituting for c, put-call parity gives

$$\max(c, p) = \max\left[p,\ p + S_1 e^{-q(T_2-T_1)} - K e^{-r(T_2-T_1)}\right]$$

$$= p + \max\left[0,\ S_1 e^{-q(T_2-T_1)} - K e^{-r(T_2-T_1)}\right]$$

This shows that the chooser option can be decomposed into

1. A put option with strike price K and maturity T_2; and
2. $e^{-q(T_2-T_1)}$ call options with strike price $Ke^{-(r-q)(T_2-T_1)}$ and maturity T_1.

Problem 24.6.

Section 24.6 gives two formulas for a down-and-out call. The first applies to the situation where the barrier, H, is less than or equal to the strike price, K. The second applies to the situation where $H \geq K$. Show that the two formulas are the same when $H = K$.

Consider the formula for c_{do} when $H \geq K$

$$c_{\text{do}} = S_0 N(x_1)e^{-qT} - Ke^{-rT}N(x_1 - \sigma\sqrt{T}) - S_0 e^{-qT}(H/S_0)^{2\lambda}N(y_1)$$

$$+Ke^{-rT}(H/S_0)^{2\lambda-2}N(y_1 - \sigma\sqrt{T})$$

Substituting $H = K$ and noting that

$$\lambda = \frac{r - q + \sigma^2/2}{\sigma^2}$$

we obtain $x_1 = d_1$ so that

$$c_{\text{do}} = c - S_0 e^{-qT}(H/S_0)^{2\lambda}N(y_1) + Ke^{-rT}(H/S_0)^{2\lambda-2}N(y_1 - \sigma\sqrt{T})$$

The formula for c_{di} when $H \leq K$ is

$$c_{\text{di}} = S_0 e^{-qT}(H/S_0)^{2\lambda}N(y) - Ke^{-rT}(H/S_0)^{2\lambda-2}N(y - \sigma\sqrt{T})$$

Since $c_{\text{do}} = c - c_{\text{di}}$

$$c_{\text{do}} = c - S_0 e^{-qT}(H/S_0)^{2\lambda}N(y) + Ke^{-rT}(H/S_0)^{2\lambda-2}N(y - \sigma\sqrt{T})$$

From the formulas in the text $y_1 = y$ when $H = K$. The two expression for c_{do} are therefore equivalent when $H = K$

Problem 24.7.

Explain why a down-and-out put is worth zero when the barrier is greater than the strike price.

The option is in the money only when the asset price is less than the strike price. However, in these circumstances the barrier has been hit and the option has ceased to exist.

Problem 24.8.

Suppose that the strike price of an American call option on a non-dividend-paying stock grows at rate g. Show that if g is less than the risk-free rate, r, it is never optimal to exercise the call early.

The argument is similar to that given in Chapter 9 for a regular option on a non-dividend-paying stock. Consider a portfolio consisting of the option and cash equal to the present value of the terminal strike price. The initial cash position is

$$Ke^{gT-rT}$$

By time τ $(0 \leq \tau \leq T)$, the cash grows to

$$Ke^{-r(T-\tau)+gT} = Ke^{g\tau}e^{-(r-g)(T-\tau)}$$

Since $r > g$, this is less than $Ke^{g\tau}$ and therefore is less than the amount required to exercise the option. It follows that, if the option is exercised early, the terminal value of the portfolio is less than S_T. At time T the cash balance is Ke^{gT}. This is exactly what is required to exercise the option. If the early exercise decision is delayed until time T, the terminal value of the portfolio is therefore

$$\max[S_T,\, Ke^{gT}]$$

This is at least as great as S_T. It follows that early exercise cannot be optimal.

Problem 24.9.

How can the value of a forward start put option on a non-dividend-paying stock be calculated if it is agreed that the strike price will be 10% greater than the stock price at the time the option starts?

When the strike price of an option on a non-dividend-paying stock is defined as 10% greater that the stock price, the value of the option is proportional to the stock price. The same argument as that given in the text for forward start options shows that if t_1 is the time when the option starts and t_2 is the time when it finishes, the option has the same value as an option starting today with a life of $t_2 - t_1$ and a strike price of 1.1 times the current stock price.

Problem 24.10.

If a stock price follows geometric Brownian motion, what process does $A(t)$ follow where $A(t)$ is the arithmetic average stock price between time zero and time t?

Assume that we start calculating averages from time zero. The relationship between $A(t + \Delta t)$ and $A(t)$ is

$$A(t + \Delta t) \times (t + \Delta t) = A(t) \times t + S(t) \times \Delta t$$

where $S(t)$ is the stock price at time t and terms of higher order than Δt are ignored. If we continue to ignore terms of higher order than Δt, it follows that

$$A(t+\Delta t) = A(t)\left[1-\frac{\Delta t}{t}\right] + S(t)\frac{\Delta t}{t}$$

Taking limits as Δt tends to zero

$$dA(t) = \frac{S(t)-A(t)}{t}dt$$

The process for $A(t)$ has a stochastic drift and no dz term. The process makes sense intuitively. Once some time has passed, the change in S in the next small portion of time has only a second order effect on the average. If S equals A the average has no drift; if $S > A$ the average is drifting up; if $S < A$ the average is drifting down.

Problem 24.11.

Explain why delta hedging is easier for Asian options than for regular options.

In an Asian option the payoff becomes more certain as time passes and the delta always approaches zero as the maturity date is approached. This makes delta hedging easy. Barrier options cause problems for delta hedgers when the asset price is close to the barrier because delta is discontinuous.

Problem 24.12.

Calculate the price of a one-year European option to give up 100 ounces of silver in exchange for one ounce of gold. The current prices of gold and silver are \$380 and \$4, respectively; the risk-free interest rate is 10% per annum; the volatility of each commodity price is 20%; and the correlation between the two prices is 0.7. Ignore storage costs.

The value of the option is given by the formula in the text

$$V_0e^{-q_2T}N(d_1) - U_0e^{-q_1T}N(d_2)$$

where

$$d_1 = \frac{\ln(V_0/U_0) + (q_1 - q_2 + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

and

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

In this case, $V_0 = 380$, $U_0 = 400$, $q_1 = 0$, $q_2 = 0$, $T = 1$, and

$$\sigma = \sqrt{0.2^2 + 0.2^2 - 2\times 0.7\times 0.2\times 0.2} = 0.1549$$

Because $d_1 = -0.2537$ and $d_2 = -0.4086$, the option price is

$$380N(-0.2537) - 400N(-0.4086) = 15.38$$

or \$15.38.

Problem 24.13.

Is a European down-and-out option on an asset worth the same as a European down-and-out option on the asset's futures price for a futures contract maturing at the same time as the option?

No. If the future's price is above the spot price during the life of the option, it is possible that the spot price will hit the barrier when the futures price does not.

Problem 24.14.

Answer the following questions about compound options

(a) What put–call parity relationship exists between the price of a European call on a call and a European put on a call? Show that the formulas given in the text satisfy the relationship.

(b)What put–call parity relationship exists between the price of a European call on a put and a European put on a put? Show that the formulas given in the text satisfy the relationship.

(a) The put–call relationship is

$$cc + K_1 e^{-rT_1} = pc + c$$

where cc is the price of the call on the call, pc is the price of the put on the call, c is the price today of the call into which the options can be exercised at time T_1, and K_1 is the exercise price for cc and pc. The proof is similar to that in Chapter 9 for the usual put–call parity relationship. Both sides of the equation represent the values of portfolios that will be worth max(c, K_1) at time T_1. Because

$$M(a, b; \rho) = N(a) - M(a, -b; -\rho) = N(b) - M(-a, b, ; -\rho)$$

and

$$N(x) = 1 - N(-x)$$

we obtain

$$cc - pc = Se^{-qT_2}N(b_1) - K_2e^{-rT_2}N(b_2) - K_1e^{-rT_1}$$

Since

$$c = Se^{-qT_2}N(b_1) - K_2e^{-rT_2}N(b_2)$$

put–call parity is consistent with the formulas

(b) The put–call relationship is

$$cp + K_1 e^{-rT_1} = pp + p$$

where cp is the price of the call on the put, pp is the price of the put on the put, p is the price today of the put into which the options can be exercised at time T_1, and K_1 is the exercise price for cp and pp. The proof is similar to that in Chapter 9 for the usual put–call parity relationship. Both sides of the equation represent the values of portfolios that will be worth $\max(p, K_1)$ at time T_1. Because

$$M(a, b; \rho) = N(a) - M(a, -b; -\rho) = N(b) - M(-a, b, ; -\rho)$$

and

$$N(x) = 1 - N(-x)$$

it follows that

$$cp - pp = -Se^{-qT_2}N(-b_1) + K_2e^{-rT_2}N(-b_2) - K_1e^{-rT_1}$$

Because

$$p = -Se^{-qT_2}N(-b_1) + K_2e^{-rT_2}N(-b_2)$$

put–call parity is consistent with the formulas.

Problem 24.15.

Does a floating lookback call become more valuable or less valuable as we increase the frequency with which we observe the asset price in calculating the minimum?

As we increase the frequency we observe a more extreme minimum which increases the value of a lookback call.

Problem 24.16.

Does a down-and-out call become more valuable or less valuable as we increase the frequency with which we observe the asset price in determining whether the barrier has been crossed? What is the answer to the same question for a down-and-in call?

As we increase the frequency with which the asset price is observed, the asset price becomes more likely to hit the barrier and the value of a down-and-out call goes down. For a similar reason the value of a down-and-in call goes up. The adjustment mentioned in the text, suggested by Broadie, Glasserman, and Kou, moves the barrier further out as the asset price is observed less frequently. This increases the price of a down-and-out option and reduces the price of a down-and-in option.

Problem 24.17.

Explain why a regular European call option is the sum of a down-and-out European call and a down-and-in European call. Is the same true for American call options?

If the barrier is reached the down-and-out option is worth nothing while the down-and-in option has the same value as a regular option. If the barrier is not reached the down-and-in option is worth nothing while the down-and-out option has the same value as a regular option. This is why a down-and-out call option plus a down-and-in call option is worth the same as a regular option. A similar argument cannot be used for American options.

Problem 24.18.

What is the value of a derivative that pays off \$100 in six months if the S&P 500 index is greater than 1,000 and zero otherwise? Assume that the current level of the index is 960, the risk-free rate is 8% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 20%.

This is a cash-or-nothing call. The value is $100N(d_2)e^{-0.08\times 0.5}$ where

$$d_2 = \frac{\ln(960/1000) + (0.08 - 0.03 - 0.2^2/2) \times 0.5}{0.2 \times \sqrt{0.5}} = -0.1826$$

Since $N(d_2) = 0.4276$ the value of the derivative is \$41.08.

Problem 24.19.

In a three-month down-and-out call option on silver futures the strike price is \$20 per ounce and the barrier is \$18. The current futures price is \$19, the risk-free interest rate is 5%, and the volatility of silver futures is 40% per annum. Explain how the option works and calculate its value. What is the value of a regular call option on silver futures with the same terms? What is the value of a down-and-in call option on silver futures with the same terms?

This is a regular call with a strike price of \$20 that ceases to exist if the futures price hits \$18. With the notation in the text $H = 18$, $K = 20$, $S = 19$, $r = 0.05$, $\sigma = 0.4$, $q = 0.05$, $T = 0.25$. From this $\lambda = 0.5$ and

$$y = \frac{\ln[18^2/(19 \times 20)]}{0.4\sqrt{0.25}} + 0.5 \times 0.4\sqrt{0.25} = -0.69714$$

The value of a down-and-out call plus a down-and-in call equals the value of a regular call. Substituting into the formula given when $H < K$ we get $c_{\text{di}} = 0.4638$. The regular Black–Scholes formula gives $c = 1.0902$. Hence $c_{\text{do}} = 0.6264$. (These answers can be checked with DerivaGem.)

Problem 24.20.

A new European-style floating lookback call option on a stock index has a maturity of nine months. The current level of the index is 400, the risk-free rate is 6% per annum, the dividend yield on the index is 4% per annum, and the volatility of the index is 20%. Use DerivaGem to value the option.

DerivaGem shows that the value is 53.38. Note that the Minimum to date and Maximum to date should be set equal to the current value of the index for a new deal. (See material on DerivaGem at the end of the book.)

Problem 24.21.

Estimate the value of a new six-month European-style average price call option on a non-dividend-paying stock. The initial stock price is \$30, the strike price is \$30, the risk-free interest rate is 5%, and the stock price volatility is 30%.

We can use the analytic approximation given in the text.

$$M_1 = \frac{(e^{0.05\times 0.5} - 1) \times 30}{0.05 \times 0.5} = 30.378$$

Also $M_2 = 936.9$ so that $\sigma = 17.41\%$. The option can be valued as a futures option with $F_0 = 30.378$, $K = 30$, $r = 5\%$, $\sigma = 17.41\%$, and $t = 0.5$. The price is 1.637.

Problem 24.22.

Use DerivaGem to calculate the value of:

a. *A regular European call option on a non-dividend-paying stock where the stock price is \$50, the strike price is \$50, the risk-free rate is 5% per annum, the volatility is 30%, and the time to maturity is one year.*
b. *A down-and-out European call which is as in (a) with the barrier at \$45.*
c. *A down-and-in European call which is as in (a) with the barrier at \$45.*

Show that the option in (a) is worth the sum of the values of the options in (b) and (c).

(a) The price of a regular European call option is 7.116.
(b) The price of the down-and-out call option is 4.696.
(c) The price of the down-and-in call option is 2.419.

The price of a regular European call is the sum of the prices of down-and-out and down-and-in options.

Problem 24.23.

Explain adjustments that have to be made when $r = q$ for a) the valuation formulas for lookback call options in Section 24.8 and b) the formulas for M_1 and M_2 in Section 24.10.

When $r = q$ in the expression for a lookback call in Section 24.8 $a_1 = a_3$ and $Y_1 = \ln(S_0/S_{\min})$ so that the expression for a lookback call becomes

$$S_0 e^{-qT} N(a_1) - S_{\min} e^{-rT} N(a_2)$$

As q approaches r in Section 24.10 we get

$$M_1 = S_0$$

$$M_2 = \frac{2e^{\sigma^2 T} S_0^2}{\sigma^4 T^2} - \frac{2S_0^2}{T^2}\frac{1+\sigma^2 T}{\sigma^4}$$

Problem 24.24.

Value the variance swap in Example 24.3 of Section 24.13 assuming that the implied volatilities for options with strike prices 800, 850, 900, 950, 1,000, 1,050, 1,100, 1,150, 1,200 are 20%, 20.5%, 21%, 21.5%, 22%, 22.5%, 23%, 23.5%, 24%, respectively.

In this case, DerivaGem shows that $Q(K_1) = 0.1772$,$Q(K_2) = 1.1857$, $Q(K_3) = 4.9123$, $Q(K_4) = 14.2374$, $Q(K_5) = 45.3738$, $Q(K_6) = 35.9243$, $Q(K_7) = 20.6883$, $Q(K_8) = 11.4135$, $Q(K_9) = 6.1043$. $\hat{E}(\bar{V}) = 0.0502$. The value of the variance swap is \$0.51 million.

ASSIGNMENT QUESTIONS

Problem 24.25.

What is the value in dollars of a derivative that pays off £10,000 in one year provided that the dollar–sterling exchange rate is greater than 1.5000 at that time? The current exchange rate is 1.4800. The dollar and sterling interest rates are 4% and 8% per annum respectively. The volatility of the exchange rate is 12% per annum.

It is instructive to consider two different ways of valuing this instrument. From the perspective of a sterling investor it is a cash or nothing put. The variables are $S_0 = 1/1.48 = 0.6757$, $K = 1/1.50 = 0.6667$, $r = 0.08$, $q = 0.04$, $\sigma = 0.12$, and $T = 1$. The derivative pays off if the exchange rate is less than 0.6667. The value of the derivative is $10,000N(-d_2)e^{-0.08\times 1}$ where

$$d_2 = \frac{\ln(0.6757/0.6667) + (0.08 - 0.04 - 0.12^2/2)}{0.12} = 0.3852$$

Since $N(-d_2) = 0.3501$, the value of the derivative is $10,000 \times 0.3501 \times e^{-0.08} = 3,231$ or 3,231. In dollars this is $3,231 \times 1.48 = \$4782$

From the perspective of a dollar investor the derivative is an asset or nothing call. The variables are $S_0 = 1.48$, $K = 1.50$, $r = 0.04$, $q = 0.08$, $\sigma = 0.12$ and $T = 1$. The value is $10,000N(d_1)e^{-0.08\times 1}$ where

$$d_1 = \frac{\ln(1.48/1.50) + (0.04 - 0.08 + 0.12^2/2)}{0.12} = -0.3852$$

$N(d_1) = 0.3500$ and the value of the derivative is as before $10,000 \times 1.48 \times 0.3500 \times e^{-0.08} = 4782$ or \$4,782.

Problem 24.26.

Consider an up-and-out barrier call option on a non-dividend-paying stock when the stock price is 50, the strike price is 50, the volatility is 30%, the risk-free rate is 5%, the time to maturity is one year, and the barrier at \$80. Use the software to value the option and graph the relationship between (a) the option price and the stock price, (b) the delta and the option price, (c) the option price and the time to maturity, and (d) the option price and the volatility. Provide an intuitive explanation for the results you get. Show that the delta, gamma, theta, and vega for an up-and-out barrier call option can be either positive or negative.

The price of the option is 3.528.

(a) The option price is a humped function of the stock price with the maximum option price occuring for a stock price of about \$57. If you could choose the stock price there would be a trade off. High stock prices give a high probability that the option will be knocked out. Low stock prices give a low potential payoff. For a stock price less than \$57 delta is positive (as it is for a regular call option); for a stock price greater that \$57, delta is negative.

(b) Delta increases up to a stock price of about 45 and then decreases. This shows that gamma can be positive or negative.

(c) The option price is a humped function of the time to maturity with the maximum option price occurring for a time to maturity of 0.5 years. This is because too long a time to maturity means that the option has a high probability of being knocked out; too short a time to maturity means that the option has a low potential payoff. For a time to maturity less than 0.5 years theta is negative (as it is for a regular call option); for a time to maturity greater than 0.5 years the theta of the option is positive.

(d) The option price is also a humped function of volatility with the maximum option price being obtained for a volatility of about 20%. This is because too high a volatility means that the option has a high probability of being knocked out; too low a volatility means that the option has a low potential payoff. For volatilities less than 20% vega is positive (as it is for a regular option); for volatilities above 20% vega is negative.

Problem 24.27.

Sample Application F in the DerivaGem Application Builder Software considers the static options replication example in Section 24.13. It shows the way a hedge can be constructed using four options (as in Section 24.13) and two ways a hedge can be constructed using 16 options.

a. *Explain the difference between the two ways a hedge can be constructed using 16 options. Explain intuitively why the second method works better.*

b. *Improve on the four-option hedge by changing Tmat for the third and fourth options.*

c. *Check how well the 16-option portfolios match the delta, gamma, and vega of the barrier option.*

(a) Both approaches use a one call option with a strike price of 50 and a maturity of 0.75. In the first approach the other 15 call options have strike prices of 60 and equally spaced times to maturity. In the second approach the other 15 call options have strike prices of 60, but the spacing between the times to maturity decreases as the maturity of the barrier option is approached. The second approach to setting times to maturity produces a better hedge. This is because the chance of the barrier being hit at time t is an increasing function of t. As t increases it therefore becomes more important to replicate the barrier at time t.

(b) By using either trial and error or the Solver tool we see that we come closest to matching the price of the barrier option when the maturities of the third and fourth options are changed from 0.25 and 0.5 to 0.39 and 0.65.

(c) To calculate delta for the two 16-option hedge strategies it is necessary to change the last argument of EPortfolio from 0 to 1 in cells L42 and X42. To calculate delta for the barrier option it is necessary to change the last argument of BarrierOption in cell F12 from 0 to 1. To calculate gamma and vega the arguments must be changed to 2 and 3, respectively. The delta, gamma, and vega of the barrier option are −0.0221, −0.0035, and −0.0254. The delta, gamma, and vega of the first 16-option portfolio are −0.0262, −0.0045, and −0.1470. The delta, gamma, and vega of the second 16-option portfolio are −0.0199, −0.0037, and −0.1449. The second of the two 16-option portfolios provides

Greek letters that are closest to the Greek letters of the barrier option. Interestingly neither of the two portfolios does particularly well on vega.

Problem 24.28

Consider a down-and-out call option on a foreign currency. The initial exchange rate is 0.90, the time to maturity is two years, the strike price is 1.00, the barrier is 0.80, the domestic risk-free interest rate is 5%, the foreign risk-free interest rate is 6%, and the volatility is 25% per annum. Use DerivaGem to develop a static option replication strategy involving five options.

A natural approach is to attempt to replicate the option with positions in:

(a) A European call option with strike price 1.00 maturing in two years
(b) A European put option with strike price 0.80 maturing in two years
(c) A European put option with strike price 0.80 maturing in 1.5 years
(d) A European put option with strike price 0.80 maturing in 1.0 years
(e) A European put option with strike price 0.80 maturing in 0.5 years

The first option can be used to match the value of the down-and-out-call for $t = 2$ and $S > 1.00$. the others can be used to match it at the following (t, S) points: (1.5, 0.80) (1.0, 0.80), (0.5, 0.80), (0.0, 0.80). Following the procedure in the text, we find that the required positions in the options are as shown in the following table.

Option Type	Strike Price	Maturity (years)	Position
Call	1.00	2.00	+1.0000
Put	0.80	2.00	−0.1255
Put	0.80	1.50	−0.1758
Put	0.80	1.00	−0.0956
Put	0.80	0.50	−0.0547

The value of the portfolio initially is 0.482. This is only a little less than the value of the down-and-out-option which is 0.488. This example is different from the example in the text in a number of ways. Put options and call options are used in the replicating portfolio. The value of the replicating portfolio converges to the value of the option from below rather than from above. Also, even with relatively few options, the value of the replicating portfolio is close to the value of the down-and-out option.

Problem 24.29.

Suppose that a stock index is currently 900. The dividend yield is 2%, the risk-free rate is 5%, and the volatility is 40%. Use the results in the appendix to calculate the value of a one-year average price call where the strike price is 900 and the index level is observed at the end of each quarter for the purposes of the averaging. Compare this with the price calculated by DerivaGem for a one-year average price option where the price is observed continuously. Provide an intuitive explanation for any differences between the prices.

In this case $M_1 = 917.07$ and $M_2 = 904028.7$ so that the option can be valued as an option on futures where the futures price is 917.07 and volatility is $\sqrt{\ln(904028.7/917.07^2}$ or 26.88%. The value of the option is 100.74. DerivaGem gives the price as 86.77 (set option type =Asian). The higher price for the first option arises because the average is calculated from prices at times 0.25, 0.50, 0.75, and 1.00. The mean of these times is 0.625. By contrast the corresponding mean when the price is observed continuously is 0.50. The later a price is observed the more uncertain it is and the more it contributes to the value of the option.

Problem 24.30.

Use the DerivaGem Application Builder software to compare the effectiveness of daily delta hedging for (a) the option considered in Tables 17.2 and 17.3 and (b) an average price call with the same parameters. Use Sample Application C. For the average price option you will find it necessary to change the calculation of the option price in cell C16, the payoffs in cells H15 and H16, and the deltas (cells G46 to G186 and N46 to N186). Carry out 20 Monte Carlo simulation runs for each option by repeatedly pressing F9. On each run record the cost of writing and hedging the option, the volume of trading over the whole 20 weeks and the volume of trading between weeks 11 and 20. Comment on the results.

For the regular option the theoretical price is 239,599. For the average price option the theoretical price is 115,259. My 20 simulation runs (40 outcomes because of the antithetic calculations) gave results as shown in the following table.

	Regular Call	Ave Price Call
Ave. Hedging Cost	247,628	114,837
S.D. Hedging Cost	17,833	12,123
Ave Trading Vol (20 wks)	412,440	291,237
Ave Trading Vol (last 10 wks)	187,074	51,658

These results show that the standard deviation of using delta hedging for an average price option is lower than that for a regular option. However, using the criterion in Chapter 17 (standard deviation divided by value of option) hedge performance is better for the regular option. Hedging the average price option requires less trading, particularly in the last 10 weeks. This is because we become progressively more certain about the average price as the maturity of the option is approached.

Problem 24.31

In the DerivaGem Application Builder Software modify Sample Application D to test the effectiveness of delta and gamma hedging for a call on call compound option on a 100,000 units of a foreign currency where the exchange rate is 0.67, the domestic risk-free rate is 5%, the foreign risk-free rate is 6%, the volatility is 12%. The time to maturity of *the first option is 20 weeks, and the strike price of the first option is 0.015. The second option matures 40 weeks from today and has a strike price of 0.68. Explain how you modified the cells. Comment on hedge effectiveness.*

The value of the option is 1093. It is necessary to change cells F20 and F46 to 0.67. Cells G20 to G39 and G46 to G65 must be changed to calculate delta of the compound option. Cells H20 to H39 and H46 to H65 must be changed to calculate gamma of the compound option. Cells I20 to I40 and I46 to I66 must be changed to calculate the Black–Scholes price of the call option expiring in 40 weeks. Similarly cells J20 to J40 and J46 to J66 must be changed to calculate the delta of this option; cells K20 to K40 and K46 to K66 must be changed to calculate the gamma of the option. The payoffs in cells N9 and N10 must be calculated as MAX(I40-0.015,0)*100000 and MAX(I66-0.015,0)*100000. Delta plus gamma hedging works relatively poorly for the compound option. On 20 simulation runs the cost of writing and hedging the option ranged from 200 to 2500.

Problem 24.32.

Outperformance certificates (also called "sprint certificates", "accelerator certificates", or "speeders") are offered to investors by many European banks as a way of investing in a company's stock. The initial investment equals the company's stock price, S_0. If the stock price goes up between time 0 and time T, the investor gains k times the increase at time T where k is a constant greater than 1.0. However, the stock price used to calculate the gain at time T is capped at some maximum level M. If the stock price goes down the investor's loss is equal to the decrease. The investor does not receive dividends.

a) Show that the net gain from an outperformance certificate is a package.

b) Calculate using DerivaGem the value of a one-year outperformance certificate when the stock price is 50 euros, $k = 1.5$, $M = 70$ euros, the risk-free rate is 5%, and the stock price volatility is 25%. Dividends equal to 0.5 euros are expected in 2 months, 5 month, 8 months, and 11 months.

a) The investor's gain (loss) on an initial investment of S_0 is equivalent to:

1. A long position in k one-year European call options on the stock with a strike price equal to the current stock price.
2. A short position in k one-year European call options on the stock with a strike price equal to M
3. A short position in one European one-year put option on the stock with a strike price equal to the current stock price.

b) In this case the value of the three parts to the gain are

1. $1.5 \times 5.0056 = 7.5084$
2. $-1.5 \times 0.6339 = 0.9509$
3. -4.5138

The total value of the gain is $7.5084 - 0.9509 - 4.5138 = 2.0437$

Problem 24.33.

Carry out the analysis in Example 24.3 of Section 24.13 to value the variance swap on the assumption that the life of the swap is 1 month rather than 3 months.

In this case, $F_0 = 1022.55$ and DerivaGem shows that $Q(K_1) = 0.0366$, $Q(K_2) = 0.2858$, $Q(K_3) = 1.5822$, $Q(K_4) = 6.3708$, $Q(K_5) = 30.3864$, $Q(K_6) = 16.9233$, $Q(K_7) = 4.8180$, $Q(K_8) = 0.8639$, and $Q_9 = 0.0863$. $\hat{E}(\bar{V}) = 0.0661$. The value of the variance swap is \$2.09 million.

CHAPTER 25
Weather, Energy, and Insurance Derivatives

Notes for the Instructor

The chapter describes a number of nontraditional derivatives products. The market for weather derivatives is relatively new and quite small. By contrast, energy derivatives (especially oil derivatives) have been around for a long time. It is interesting to talk about the different ways in which oil, gas, and electricity derivatives products are structured to accommodate the different properties of these energy sources. The section on insurance derivatives provides a thumbnail sketch of how the reinsurance industry works and explains some of the derivative products that have been developed as alternates to traditional forms of reinsurance.

Students enjoy the material in this chapter and find the descriptions of how the different products work fascinating. I like to distinguish between the historical data approach to valuation and the risk-neutral valuation approach. (See the first section of the chapter.) When underlying market variables have no systematic risk (as they often do in the derivatives considered in this chapter) the historical data and risk-neutral valuation approaches are equivalent.

Students enjoy the material in this chapter and find the descriptions of how the different products work fascinating. Problem 25.15 can be used as an assignment question or for class discussion.

QUESTIONS AND PROBLEMS

Problem 25.1.

What is meant by HDD and CDD?

A day's HDD is $\max(0, 65 - A)$ and a day's CDD is $\max(0, A - 65)$ where A is the average of the highest and lowest temperature during the day at a specified weather station, measured in degrees Fahrenheit.

Problem 25.2.

How is a typical natural gas forward contract structured?

It is an agreement by one side to delivery a specified amount of gas at a roughly uniform rate during a month to a particular hub for a specified price.

Problem 25.3.

Distinguish between the historical data and the risk-neutral approach to valuing a derivative. Under what circumstance do they give the same answer.

The historical data approach to valuing an option involves calculating the expected payoff using historical data and discounting the payoff at the risk-free rate. The risk-neutral approach involves calculating the expected payoff in a risk-neutral world and discounting at the risk-free rate. The two approaches give the same answer when percentage changes in the underlying market variables have zero correlation with stock market returns. (In these circumstances all risks can be diversified away.)

Problem 25.4.

Suppose that each day during July the minimum temperature is 68° Fahrenheit and the maximum temperature is 82° Fahrenheit. What is the payoff from a call option on the cumulative CDD during July with a strike of 250 and a payment rate of $5,000 per degree day?

The average temperature each day is 75°. The CDD each day is therefore 10 and the cumulative CDD for the month is $10 \times 31 = 310$. The payoff from the call option is therefore $(310 - 250) \times 5,000 = \$300,000$.

Problem 25.5.

Why is the price of electricity more volatile than that of other energy sources?

Unlike most commodities electricity cannot be stored easily. If the demand for electricity exceeds the supply, as it sometimes does during the air conditioning season, the price of electricity in a deregulated environment will skyrocket. When supply and demand become matched again the price will return to former levels.

Problem 25.6.

Why is the historical data approach appropriate for pricing a weather derivatives contract and a CAT bond?

There is no systematic risk (i.e., risk that is priced by the market) in weather derivatives and CAT bonds.

Problem 25.7.

"HDD and CDD can be regarded as payoffs from options on temperature." Explain

HDD is $\max(65 - A,\ 0)$ where A is the average of the maximum and minimum temperature during the day. This is the payoff from a put option on A with a strike price of 65. CDD is $\max(A - 65,\ 0)$. This is the payoff from call option on A with a strike price of 65.

Problem 25.8.

Suppose that you have 50 years of temperature data at your disposal. Explain carefully the analyses you would carry out to value a forward contract on the cumulative CDD for a particular month.

It would be useful to calculate the cumulative CDD for July of each year of the last 50 years. A linear regression relationship

$$\text{CDD} = a + bt + e$$

could then be estimated where a and b are constants, t is the time in years measured from the start of the 50 years, and e is the error. This relationship allows for linear trends in temperature through time. The expected CDD for next year (year 51) is then $a + 51b$. This could be used as an estimate of the forward CDD.

Problem 25.9.

Would you expect the volatility of the one-year forward price of oil to be greater than or less than the volatility of the spot price. Explain.

The volatility of the three-month forward price will be less than the volatility of the spot price. This is because, when the spot price changes by a certain amount, mean reversion will cause the forward price will change by a lesser amount.

Problem 25.10.

What are the characteristics of an energy source where the price has a very high volatility and a very high rate of mean reversion? Give an example of such an energy source.

The price of the energy source will show big changes, but will be pulled back to its long-run average level fast. Electricity is an example of an energy source with these characteristics.

Problem 25.11.

How can an energy producer user derivative markets to hedge risks?

The energy producer faces quantity risks and price risks. It can use weather derivatives to hedge the quantity risks and energy derivatives to hedge against the price risks.

Problem 25.12.

Explain how a 5×8 option contract for May 2009 on electricity with daily exercise works. Explain how a 5×8 option contract for May 2009 on electricity with monthly exercise works. Which is worth more?

A 5×8 contract for May, 2009 is a contract to provide electricity for five days per week during the off-peak period (11PM to 7AM). When daily exercise is specified, the holder of the option is able to choose each weekday whether he or she will buy electricity at the strike price at the agreed rate. When there is monthly exercise, he or she chooses once at the beginning of the month whether electricity is to be bought at the strike price at the agreed rate for the whole month. The option with daily exercise is worth more.

Problem 25.13.

Explain how CAT bonds work.

CAT bonds (catastrophe bonds) are an alternative to reinsurance for an insurance company that has taken on a certain catastrophic risk (e.g., the risk of a hurricane or an earthquake) and wants to get rid of it. CAT bonds are issued by the insurance company. They provide a higher rate of interest than government bonds. However, the bondholders agree to forego interest, and possibly principal, to meet any claims against the insurance company that are within a prespecified range.

Problem 25.14.

Consider two bonds that have the same coupon, time to maturity and price. One is a B-rated corporate bond. The other is a CAT bond. An analysis based on historical data shows that the expected losses on the two bonds in each year of their life is the same. Which bond would you advise a portfolio manager to buy and why?

The CAT bond has very little systematic risk. Whether a particular type of catastrophe occurs is independent of the return on the market. The risks in the CAT bond are likely to be largely "diversified away" by the other investments in the portfolio. A B-rated bond does have systematic risk that cannot be diversified away. It is likely therefore that the CAT bond is a better addition to the portfolio.

ASSIGNMENT QUESTIONS

Problem 25.15.

(a) *The losses in millions of dollars are approximately*

$$\phi(150, 50^2)$$

The reinsurance contract would pay out 60% of the losses. The payout from the reinsurance contract is therefore

$$\phi(90, 30^2)$$

The cost of the reinsurance is the expected payout in a risk-neutral world, discounted at the risk-free rate. In this case, the expected payout is the same in a risk-neutral world as it is in the real world. The value of the reinsurance contract is therefore

$$90e^{-0.05\times 1} = 85.61$$

(b) *The probability that losses will be greater than \$200 million is the probability that a normally distributed variable is greater than one standard deviation above the mean. This is 0.1587. The expected payoff in millions of dollars is therefore* $0.1587 \times 100 = 15.87$ *and the value of the contract is*

$$15.87e^{-0.05\times 1} = 15.10$$

CHAPTER 26

More on Models and Numerical Procedures

Notes for the Instructor

This chapter covers a number of nonstandard procedures that can be used to value derivatives. Each section of the chapter is independent of each other section. This means that instructors can teach some sections and omit others if they want to. The first three sections expose the student to some of the alternatives to Black-Scholes that provide a better fit than Black–Scholes to the volatility smiles that are encountered in practice (see Chapter 18). Section 26.4 deals with convertibles. (This material was moved to this chapter from the chapter on credit derivatives.) Sections 26.5 to 26.7 deal with a variety of numerical procedures for handling path-dependent options, barrier options, and options involving two correlated asset prices. The final section uses examples to explain two ways of valuing American options with Monte Carlo simulation.

Problems 26.19 to 26.23 all work well as assignments.

QUESTIONS AND PROBLEMS

Problem 26.1.

Confirm that the CEV model formulas satisfy put–call parity.

It follows immediately from the equations in Section 26.1 that

$$p - c = Ke^{-rT} - S_0e^{-qT}$$

in all cases.

Problem 26.2.

Explain how you would use Monte Carlo simulation to sample paths for the asset price when Merton's jump diffusion model is used.

The probability of N jumps in time Δt is

$$\frac{e^{-\lambda\Delta t}(\lambda\Delta t)^N}{N!}$$

When Δt is small we can ignore terms of order $(\Delta t)^2$ and higher so that the probability of no jumps is $1 - \lambda\Delta t$ and the probability of one jump is $\lambda\Delta t$. During each time step of length Δt we first sample a random number between 0 and 1 to determine whether a jump takes place. Suppose for example that $\lambda = 0.8$ and $\Delta t = 0.1$ so that the probability of no jumps is 0.92 and the probability of one jump is 0.08. If the random number is between 0 and 0.92 there is no jump; if it is between 0.92 and 1, there is one jump. If there is a jump we sample from the appropriate distribution to determine the size of the jump. The change in the asset price in time Δt is then given by

$$\frac{\Delta S}{S} = (\mu - \lambda k)\Delta t + \sigma\epsilon\sqrt{\Delta t} + Q$$

where $Q = 0$ if there is no jump and Q is the size of the jump if a jump takes place. We can adjust this procedure to sample $\ln S$ rather than S and to allow for more than one jump in time Δt.

Problem 26.3.

Confirm that Merton's jump diffusion model satisfies put–call parity when the jump size is lognormal.

With the notation in the text the value of a call option, c is

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} c_n$$

where c_n is the Black–Scholes price of a call option where the variance rate is

$$\sigma^2 + \frac{ns^2}{T}$$

and the risk-free rate is

$$r - \lambda k + \frac{n\gamma}{T}$$

where $\gamma = \ln(1 + k)$. Similarly the value of a put option p is

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} p_n$$

where p_n is the Black–Scholes price of a put option with this variance rate and risk-free rate. It follows that

$$p - c = \sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} (p_n - c_n)$$

From put–call parity

$$p_n - c_n = Ke^{(-r+\lambda k)T} e^{-n\gamma} - S_0 e^{-qT}$$

Because

$$e^{-n\gamma} = (1 + k)^{-n}$$

it follows that

$$p - c = \sum_{n=0}^{\infty} \frac{e^{-\lambda' T + \lambda k T}(\lambda' T/(1 + k))^n}{n!} Ke^{-rT} - \sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} S_0 e^{-qT}$$

Using $\lambda' = \lambda(1+k)$ this becomes

$$\frac{1}{e^{\lambda T}}\sum_{n=0}^{\infty}\frac{(\lambda T)^n}{n!}Ke^{-rT} - \frac{1}{e^{\lambda' T}}\sum_{n=0}^{\infty}\frac{(\lambda' T)^n}{n!}S_0e^{-qT}$$

From the expansion of the exponential function we get

$$e^{\lambda T} = \sum_{n=0}^{\infty}\frac{(\lambda T)^n}{n!}$$

$$e^{\lambda' T} = \sum_{n=0}^{\infty}\frac{(\lambda' T)^n}{n!}$$

Hence

$$p - c = Ke^{-rT} - S_0e^{-qT}$$

showing that put–call parity holds.

Problem 26.4.

Suppose that the volatility of an asset will be 20% from month 0 to month 6, 22% from month 6 to month 12, and 24% from month 12 to month 24. What volatility should be used in Black–Scholes to value a two-year option?

The average variance rate is

$$\frac{6\times 0.2^2 + 6\times 0.22^2 + 12\times 0.24^2}{24} = 0.0509$$

The volatility used should be $\sqrt{0.0509} = 0.2256$ or 22.56%.

Problem 26.5.

Consider the case of Merton's jump diffusion model where jumps always reduce the asset price to zero. Assume that the average number of jumps per year is λ. Show that the price of a European call option is the same as in a world with no jumps except that the risk-free rate is $r+\lambda$ rather than r. Does the possibility of jumps increase or reduce the value of the call option in this case? (Hint: Value the option assuming no jumps and assuming one or more jumps. The probability of no jumps in time T is $e^{-\lambda T}$).

In a risk-neutral world the process for the asset price exclusive of jumps is

$$\frac{dS}{S} = (r - q - \lambda k)\,dt + \sigma\,dz$$

In this case $k = -1$ so that the process is

$$\frac{dS}{S} = (r - q + \lambda)\,dt + \sigma\,dz$$

The asset behaves like a stock paying a dividend yield of $q-\lambda$. This shows that, conditional on no jumps, call price

$$S_0e^{-(q-\lambda)T}N(d_1) - Ke^{-rT}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r - q + \lambda + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

There is a probability of $e^{-\lambda T}$ that there will be no jumps and a probability of $1 - e^{-\lambda T}$ that there will be one or more jumps so that the final asset price is zero. It follows that there is a probability of $e^{-\lambda T}$ that the value of the call is given by the above equation and $1 - e^{-\lambda T}$ that it will be zero. Because jumps have no systematic risk it follows that the value of the call option is

$$e^{-\lambda T}[S_0e^{-(q-\lambda)T}N(d_1) - Ke^{-rT}]$$

or

$$S_0e^{-qT}N(d_1) - Ke^{-(r+\lambda)T}$$

This is the required result. The value of a call option is an increasing function of the risk-free interest rate (see Chapter 9). It follows that the possibility of jumps increases the value of the call option in this case.

Problem 26.6.

At time zero the price of a non-dividend-paying stock is S_0. Suppose that the time interval between 0 and T is divided into two subintervals of length t_1 and t_2. During the first subinterval, the risk-free interest rate and volatility are r_1 and σ_1, respectively. During the second subinterval, they are r_2 and σ_2, respectively. Assume that the world is risk neutral.

(a) *Use the results in Chapter 13 to determine the stock price distribution at time T in terms of r_1, r_2, σ_1, σ_2, t_1, t_2, and S_0.*
(b) *Suppose that $\overline{r}$ is the average interest rate between time zero and T and that $\overline{V}$ is the average variance rate between times zero and T. What is the stock price distribution as a function of T in terms of $\overline{r}$, $\overline{V}$, T, and S_0?*
(c) *What are the results corresponding to (a) and (b) when there are three subintervals with different interest rates and volatilities?*
(d) *Show that if the risk-free rate, r, and the volatility, σ, are known functions of time, the stock price distribution at time T in a risk-neutral world is*

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\overline{r} - \frac{\overline{V}}{2}\right)T,\ \overline{V}T\right]$$

where $\overline{r}$ is the average value of r, $\overline{V}$ is equal to the average value of σ^2, and S_0 is the stock price today.

(a) Suppose that S_1 is the stock price at time t_1 and S_T is the stock price at time T. From equation (13.3), it follows that in a risk- neutral world:

$$\ln S_1 - \ln S_0 \sim \phi\left[\left(r_1 - \frac{\sigma_1^2}{2}\right)t_1,\, \sigma_1^2 t_1\right]$$

$$\ln S_T - \ln S_1 \sim \phi\left[\left(r_2 - \frac{\sigma_2^2}{2}\right)t_2,\, \sigma_2^2 t_2\right]$$

Since the sum of two independent normal distributions is normal with mean equal to the sum of the means and variance equal to the sum of the variances

$$\begin{aligned}\ln S_T - \ln S_0 &= (\ln S_T - \ln S_1) + (\ln S_1 - \ln S_0)\\ &\sim \phi\left(r_1 t_1 + r_2 t_2 - \frac{\sigma_1^2 t_1}{2} - \frac{\sigma_2^2 t_2}{2}, \sigma_1^2 t_1 + \sigma_2^2 t_2\right)\end{aligned}$$

(b) Because

$$r_1 t_1 + r_2 t_2 = \overline{r}T$$

and

$$\sigma_1^2 t_1 + \sigma_2^2 t_2 = \overline{V}T$$

it follows that:

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\overline{r} - \frac{\overline{V}}{2}\right)T, \overline{V}T\right]$$

(c) If σ_i and r_i are the volatility and risk-free interest rate during the ith subinterval ($i = 1, 2, 3$), an argument similar to that in (a) shows that:

$$\ln S_T - \ln S_0 \sim \phi\left(r_1 t_1 + r_2 t_2 + r_3 t_3 - \frac{\sigma_1^2 t_1}{2} - \frac{\sigma_2^2 t_2}{2} - \frac{\sigma_3^2 t_3}{2}, \sigma_1^2 t_1 + \sigma_2^2 t_2 + \sigma_3^2 t_3\right)$$

where t_1, t_2 and t_3 are the lengths of the three subintervals. It follows that the result in (b) is still true.

(d) The result in (b) remains true as the time between time zero and time T is divided into more subintervals, each having its own risk-free interest rate and volatility. In the limit, it follows that, if r and σ are known functions of time, the stock price distribution at time T is the same as that for a stock with a constant interest rate and variance rate with the constant interest rate equal to the average interest rate and the constant variance rate equal to the average variance rate.

Problem 26.7.

Write down the equations for simulating the path followed by the asset price in the stochastic volatility model in equation (26.2) and (26.3).

The equations are:

$$S(t + \Delta t) = S(t)\exp[(r - q - V(t)/2)\Delta t + \epsilon_1\sqrt{V(t)\Delta t}]$$

$$V(t + \Delta t) - V(t) = a[V_L - V(t)]\Delta t + \xi \epsilon_2 V(t)^\alpha \sqrt{\Delta t}$$

Problem 26.8.

"The IVF model does not necessarily get the evolution of the volatility surface correct." Explain this statement.

The IVF model is designed to match the volatility surface today. There is no guarantee that the volatility surface given by the model at future times will be the same as today—or that it will be even reasonable.

Problem 26.9.

"When interest rates are constant the IVF model correctly values any derivative whose payoff depends on the value of the underlying asset at only one time." Explain this statement.

The IVF model ensures that the risk-neutral probability distribution of the asset price at any future time conditional on its value today is correct (or at least consistent with the market prices of options). When a derivative's payoff depends on the value of the asset at only one time the IVF model therefore calculates the expected payoff from the asset correctly. The value of the derivative is the present value of the expected payoff. When interest rates are constant the IVF model calculates this present value correctly.

Problem 26.10.

Use a three-time-step tree to value an American lookback call option on a currency when the initial exchange rate is 1.6, the domestic risk-free rate is 5% per annum, the foreign risk-free interest rate is 8% per annum, the exchange rate volatility is 15%, and the time to maturity is 18 months. Use the approach in Section 26.5.

In this case $S_0 = 1.6$, $r = 0.05$, $r_f = 0.08$, $\sigma = 0.15$, $T = 1.5$, $\Delta t = 0.5$. This means that

$$u = e^{0.15\sqrt{0.5}} = 1.1119$$

$$d = \frac{1}{u} = 0.8994$$

$$a = e^{(0.05-0.08)\times 0.5} = 0.9851$$

$$p = \frac{a-d}{u-d} = 0.4033$$

$$1 - p = 0.5967$$

The option pays off

$$S_T - S_{\min}$$

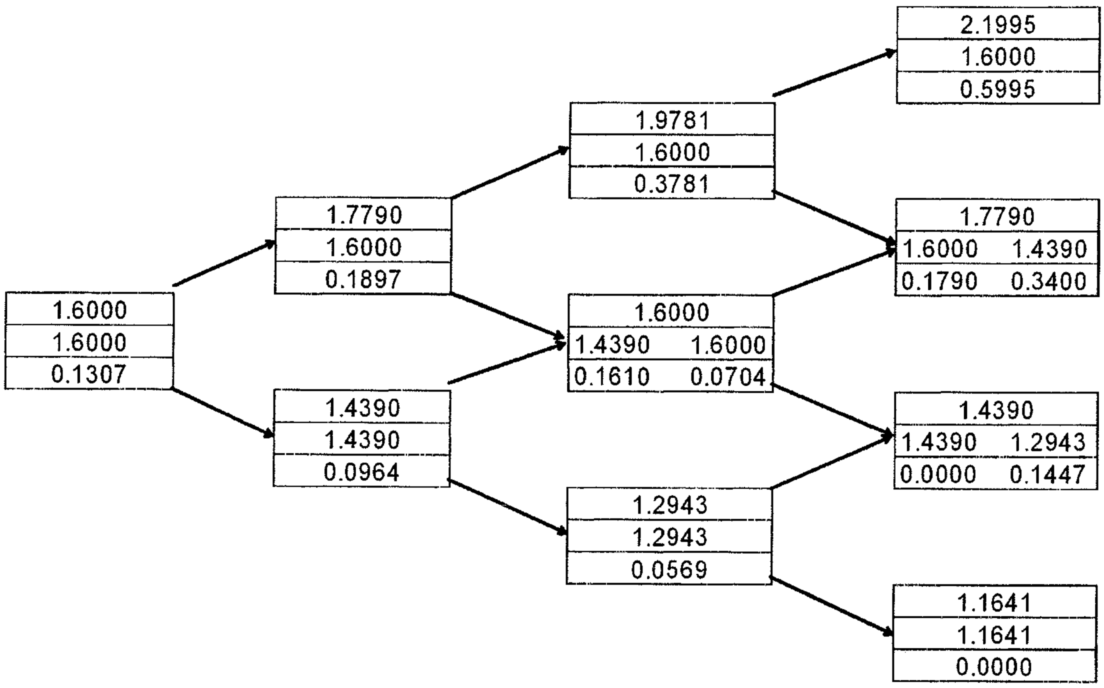


Figure S26.1 Binomial tree for Problem 26.10.

The tree is shown in Figure S26.1. At each node, the upper number is the exchange rate, the middle number(s) are the minimum exchange rate(s) so far, and the lower number(s) are the value(s) of the option. The tree shows that the value of the option today is 0.131.

Problem 26.11.

What happens to the variance-gamma model as the parameter v tends to zero?

As v tends to zero the value of g becomes T with certainty. This can be demonstrated using the GAMMADIST function in Excel. By using a series expansion for the ln function we see that ω becomes $-\theta T$. In the limit the distribution of $\ln S_T$ therefore has a mean of $\ln S_0 + (r - q)T$ and a standard deviation of $\sigma\sqrt{T}$ so that the model becomes geometric Brownian motion.

Problem 26.12.

Use a three-time-step tree to value an American put option on the geometric average of the price of a non-dividend-paying stock when the stock price is \$40, the strike price is

\$40, the risk-free interest rate is 10% per annum, the volatility is 35% per annum, and the time to maturity is three months. The geometric average is measured from today until the option matures.

In this case $S_0 = 40$, $K = 40$, $r = 0.1$, $\sigma = 0.35$, $T = 0.25$, $\Delta t = 0.08333$. This means that

$$u = e^{0.35\sqrt{0.08333}} = 1.1063$$

$$d = \frac{1}{u} = 0.9039$$

$$a = e^{0.1\times 0.08333} = 1.008368$$

$$p = \frac{a-d}{u-d} = 0.5161$$

$$1 - p = 0.4839$$

The option pays off

$$40 - \overline{S}$$

where $\overline{S}$ denotes the geometric average. The tree is shown in Figure S26.2. At each node, the upper number is the stock price, the middle number(s) are the geometric average(s), and the lower number(s) are the value(s) of the option. The geometric averages are calculated using the first, the last and all intermediate stock prices on the path. The tree shows that the value of the option today is \$1.40.

Problem 26.13.

Can the approach for valuing path dependent options in Section 26.5 be used for a two-year American-style option that provides a payoff equal to $\max(S_{\text{ave}} - K, 0)$ *where* S_{ave} *is the average asset price over the three months preceding exercise. Explain your answer.*

As mentioned in Section 26.4, for the procedure to work it must be possible to calculate the value of the path function at time $\tau + \Delta t$ from the value of the path function at time τ and the value of the underlying asset at time $\tau + \Delta t$. When S_{ave} is calculated from time zero until the end of the life of the option (as in the example considered in Section 26.4) this condition is satisfied. When it is calculated over the last three months it is not satisfied. This is because, in order to update the average with a new observation on S, it is necessary to know the observation on S from three months ago that is now no longer part of the average calculation.

Problem 26.14.

Verify that the 6.492 number in Figure 26.4 is correct.

We consider the situation where the average at node X is 53.83. If there is an up movement to node Y the new average becomes:

$$\frac{53.83 \times 5 + 54.68}{6} = 53.97$$

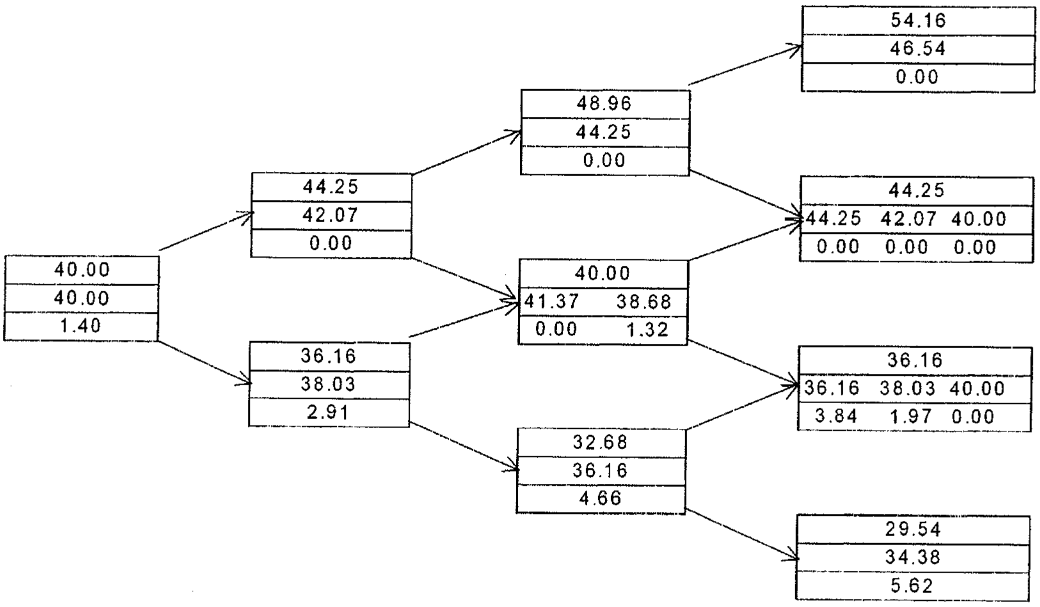


Figure S26.2 Binomial tree for Problem 26.12.

Interpolating, the value of the option at node Y when the average is 53.97 is

$$\frac{(53.97 - 51.12) \times 8.635 + (54.26 - 53.97) \times 8.101}{54.26 - 51.12} = 8.586$$

Similarly if there is a down movement the new average will be

$$\frac{53.83 \times 5 + 45.72}{6} = 52.48$$

In this case the option price is 4.416. The option price at node X when the average is 53.83 is therefore:

$$8.586 \times 0.5056 + 4.416 \times 0.4944)e^{-0.1 \times 0.05} = 6.492$$

Problem 26.15.

Examine the early exercise policy for the eight paths considered in the example in Section 26.8. What is the difference between the early exercise policy given by the least squares approach and the exercise boundary parameterization approach? Which gives a higher option price for the paths sampled?

Under the least squares approach we exercise at time $t = 1$ in paths 4, 6, 7, and 8. We exercise at time $t = 2$ for none of the paths. We exercise at time $t = 3$ for path 3. Under the exercise boundary parameterization approach we exercise at time $t = 1$ for paths 6 and 8. We exercise at time $t = 2$ for path 7. We exercise at time $t = 3$ for paths 3 and 4. For the paths sampled the exercise boundary parameterization approach gives a higher value for the option. However, it may be biased upward. As mentioned in the text, once the early exercise boundary has been determined in the exercise boundary parameterization approach a new Monte Carlo simulation should be carried out.

Problem 26.16.

Consider a European put option on a non-dividend paying stock when the stock price is \$100, the strike price is \$110, the risk-free rate is 5% per annum, and the time to maturity is one year. Suppose that the average variance rate during the life of an option has a 0.20 probability of being 0.06, a 0.5 probability of being 0.09, and a 0.3 probability of being 0.12. The volatility is uncorrelated with the stock price. Estimate the value of the option. Use DerivaGem.

If the average variance rate is 0.06, the value of the option is given by Black-Scholes with a volatility of $\sqrt{0.06} = 24.495\%$; it is 12.460. If the average variance rate is 0.09, the value of the option is given by Black-Scholes with a volatility of $\sqrt{0.09} = 30.000\%$; it is 14.655. If the average variance rate is 0.12, the value of the option is given by Black-Scholes with a volatility of $\sqrt{0.12} = 34.641\%$; it is 16.506. The value of the option is the Black-Scholes price integrated over the probability distribution of the average variance rate. It is

$$0.2 \times 12.460 + 0.5 \times 14.655 + 0.3 \times 16.506 = 14.77$$

Problem 26.17.

When there are two barriers how can a tree be designed so that nodes lie on both barriers?

Suppose that there are two horizontal barriers, H_1 and H_2, with $H_1 < H_2$ and that the underlying stock price follows geometric Brownian motion. In a trinomial tree, there are three possible movements in the asset's price at each node: up by a proportional amount u; stay the same; and down by a proportional amount d where $d = 1/u$. We can always choose u so that nodes lie on both barriers. The condition that must be satisfied by u is

$$H_2 = H_1 u^N$$

or

$$\ln H_2 = \ln H_1 + N \ln u$$

for some integer N.

When discussing trinomial trees in Section 19.4, the value suggested for u was $e^{\sigma\sqrt{3\Delta t}}$ so that $\ln u = \sigma\sqrt{3\Delta t}$. In the situation considered here, a good rule is to choose $\ln u$ as close as possible to this value, consistent with the condition given above. This means that we set

$$\ln u = \frac{\ln H_2 - \ln H_1}{N}$$

where

$$N = \text{int}\left[\frac{\ln H_2 - \ln H_1}{\sigma\sqrt{3\Delta t}} + 0.5\right]$$

and $\text{int}(x)$ is the integral part of x. This means that nodes are at values of the stock price equal to H_1, $H_1 u$, $H_1 u^2$, ..., $H_1 u^N = H_2$

Normally the trinomial stock price tree is constructed so that the central node is the initial stock price. In this case, it is unlikely that the current stock price happens to be $H_1 u^i$ for some i. To deal with this the first trinomial movement should be from the initial stock price to $H_1 u^{i-1}$, $H_1 u^i$ and $H_1 u^{i+1}$ where i is chosen so that $H_1 u^i$ is closest to the current stock price. The probabilities on all branches of the tree are chosen, as usual, to match the first two moments of the stochastic process followed by the asset price. The approach works well except when the initial asset price is close to a barrier.

Problem 26.18.

Consider an 18-month zero-coupon bond with a face value of $100 that can be converted into five shares of the company's stock at any time during its life. Suppose that the current share price is $20, no dividends are paid on the stock, the risk-free rate for all maturities is 6% per annum with continuous compounding, and the share price volatility is 25% per annum. Assume that the default intensity is 3% per year and the recovery rate is 35%. The bond is callable at $110. Use a three-time-step tree to calculate the value of the bond. What is the value of the conversion option (net of the issuer's call option)?

In this case $\Delta t = 0.5$, $\lambda = 0.03$, $\sigma = 0.25$, $r = 0.06$ and $q = 0$ so that $u = 1.1360$, $d = 0.8803$, $a = 1.0305$, $p_u = 0.6386$, $p_d = 0.3465$, and the probability on default branches is 0.0149. This leads to the tree shown in Figure S26.3. The bond is called at nodes B and D and this forces exercise. Without the call the value at node D would be 129.55, the value at node B would be 115.94, and the value at node A would be 105.18. The value of the call option to the bond issuer is therefore $105.18 - 103.72 = 1.46$.

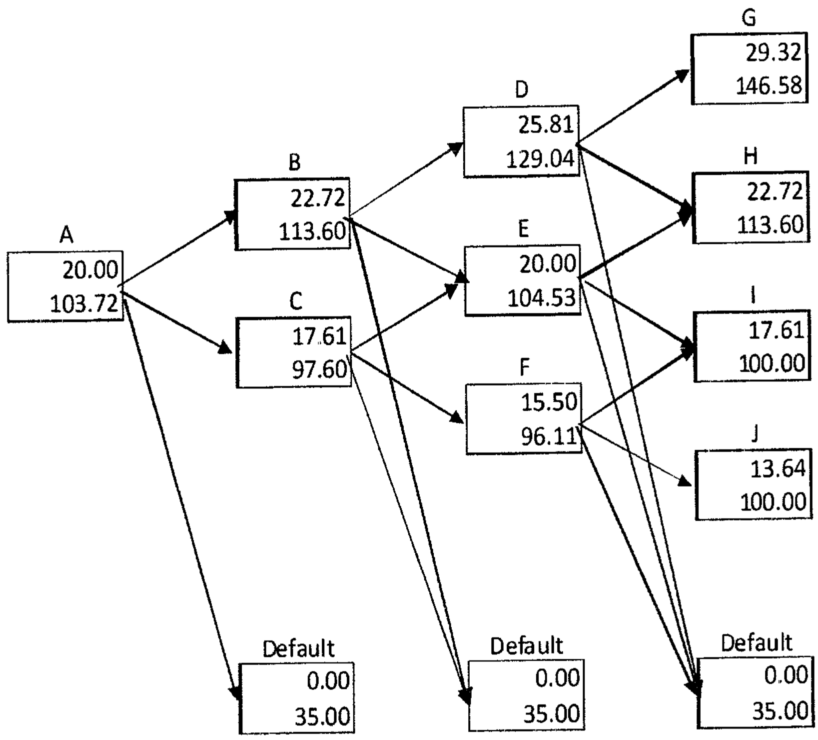


Figure S26.3 Tree for Problem 26.18

ASSIGNMENT QUESTIONS

Problem 26.19.

A new European-style lookback call option on a stock index has a maturity of nine months. The current level of the index is 400, the risk-free rate is 6% per annum, the dividend yield on the index is 4% per annum, and the volatility of the index is 20%. Use the approach in Section 26.5 to value the option and compare your answer to the result given by DerivaGem using the analytic valuation formula.

Using three-month time steps the tree parameters are $\Delta t = 0.254$, $u = 1.1052$, $d = 0.9048$, $a = 1.0050$, $p = 0.5000$. The tree is shown in Figure M26.1. The value of the lookback option is 40.47. (A more efficient procedure for giving the same result is in Technical Note 13. We construct a tree for $Y(t) = G(t)/S(t)$ where $G(t)$ is the minimum value of the index to date and $S(t)$ is the value of the index at time t. The tree is shown in Figure M26.2. It values the option in units of the stock index. This means that we value an instrument that pays off $1 - Y(t)$. The tree shows that the value of the option is

0.1019 units of the stock index or $400 \times 0.1019 = 40.47$ dollars, as given by Figure M26.1. DerivaGem shows that the value given by the analytic formula is 53.38. This is higher than the value given by the tree because the tree assumes that the stock price is observed only three times when the minimum is calculated.

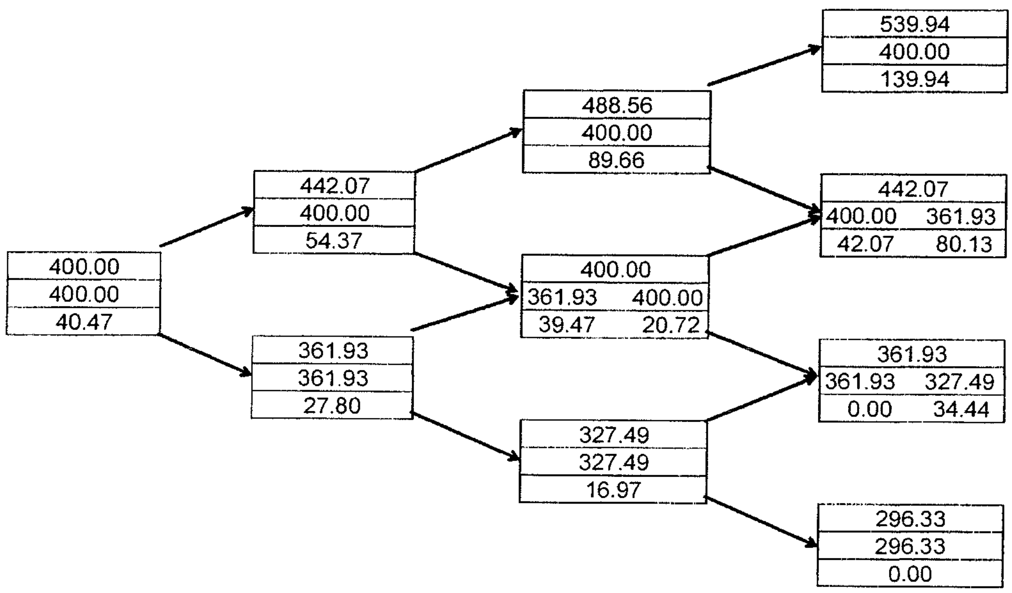


Figure M26.1 Tree for Problem 26.19.

Problem 26.20.

Suppose that the volatilities used to price a six-month currency option are as in Table 18.2. Assume that the domestic and foreign risk-free rates are 5% per annum and the current exchange rate is 1.00. Consider a bull spread that consists of a long position in a six-month call option with strike price 1.05 and a short position in a six-month call option with a strike price 1.10.

(a) What is the value of the spread?

(b) What single volatility if used for both options gives the correct value of the bull spread? (Use the DerivaGem Application Builder in conjunction with Goal Seek or Solver.)

(c) Does your answer support the assertion at the beginning of the chapter that the correct volatility to use when pricing exotic options can be counterintuitive?

(d) Does the IVF model give the correct price for the bull spread?

(a) The six-month call option with a strike price of 1.05 should be valued with a volatility of 13.4% and is worth 0.01829. The call option with a strike price of 1.10 should be

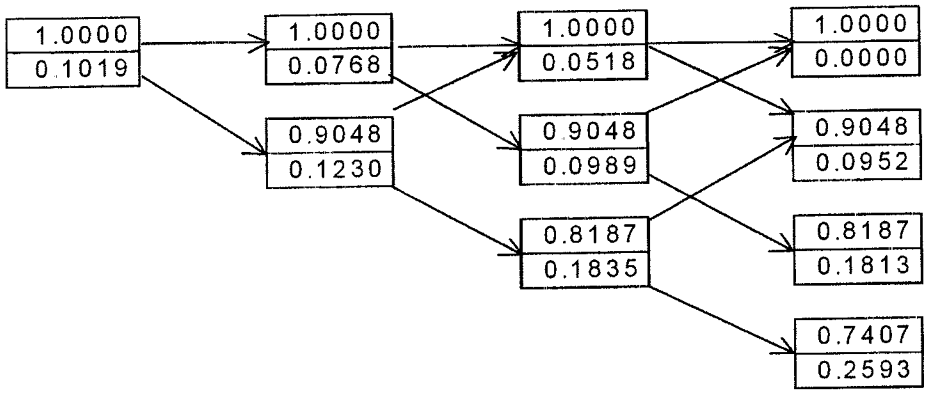


Figure M26.2 Tree for Problem 26.18 Using Results in Technical Note 13.

valued with a volatility of 14.3% and is worth 0.00959. The bull spread is therefore worth 0.01829 − 0.00959 = 0.00870.

(b) We now ask what volatility, if used to value both options, gives this price. Using the DerivaGem Application Builder in conjunction with Goal Seek we find that the answer is 11.42%.

(c) Yes, this does support the contention at the beginning of the chapter that the correct volatility for valuing exotic options can be counterintuitive. We might reasonably expect the volatility to be between 13.4% (the volatility used to value the first option) and 14.3% (the volatility used to value the second option). 11.42% is well outside this range. The reason why the volatility is relatively low is as follows. The option provides the same payoff as a regular option with a 1.05 strike price when the asset price is between 1.05 and 1.10 and a lower payoff when the asset price is over 1.10. The implied probability distribution of the asset price (see Figure 18.2) is less heavy than the lognormal distribution in the 1.05 to 1.10 range and more heavy than the lognormal distribution in the > 1.10 range. This means that using a volatility of 13.4% (which is the implied volatility of a regular option with a strike price of 1.05) will give a price than is too high.

(d) The bull spread provides a payoff at only one time. It is therefore correctly valued by the IVF model.

Problem 26.21.

Repeat the analysis in Section 26.8 for the put option example on the assumption that the strike price is 1.13. Use both the least squares approach and the exercise boundary parameterization approach.

Consider first the least squares approach. At the two-year point the option is in the money for paths 1, 3, 4, 6, and 7. The five observations on S are 1.08, 1.07, 0.97, 0.77, and 0.84. The five continuation values are 0, $0.10e^{-0.06}$, $0.21e^{-0.06}$, $0.23e^{-0.06}$, $0.12e^{-0.06}$. The best fit continuation value is

$$-1.394 + 3.795S - 2.276S^2$$

The best fit continuation values for the five paths are 0.0495, 0.0605, 0.1454, 0.1785, and 0.1876. These show that the option should be exercised at the two-year point for all five paths. There are six paths at the one-year point for which the option is in the money. These are paths 1, 4, 5, 6, 7, and 8. The six observations on S are 1.09, 0.93, 1.11, 0.76, 0.92, and 0.88. The six continuation values are $0.05e^{-0.06}$, $0.16e^{-0.06}$, 0, $0.36e^{-0.06}$, $0.29e^{-0.06}$, and 0. The best fit continuation value is

$$2.055 - 3.317S + 1.341S^2$$

The best fit continuation values for the six paths are 0.0327, 0.1301, 0.0253, 0.3088, 0.1385, and 0.1746. These show that the option should be exercised at the one-year point for paths 1, 4, 6, 7, and 8 The value of the option if not exercised at time zero is therefore

$$\frac{1}{8}(0.04e^{-0.06} + 0 + 0.06e^{-0.012} + 0.20e^{-0.06} + 0 + 0.37e^{-0.06} + 0.21e^{-0.06} + 0.25e^{-0.06})$$

or 0.133. Exercising at time zero would yield 0.13. The option should therefore not be exercised at time zero and its value is 0.133.

Consider next the exercise boundary parametrization approach. At time two years it is optimal to exercise when the stock price is 0.84 or below. At time one year it is optimal to exercise whenever the option is in the money. The value of the option assuming no early exercise at time zero is therefore

$$\frac{1}{8}(0.04e^{-0.06} + 0 + 0.10e^{-0.018} + 0.20e^{-0.06} + 0.02e^{-0.06}$$

$$+0.37e^{-0.06} + 0.21e^{-0.06} + 0.25e^{-0.06})$$

or 0.139. Exercising at time zero would yield 0.13. The option should therefore not be exercised at time zero. The value at time zero is 0.139. However, this tends to be high. As explained in the text, we should use one Monte Carlo simulation to determine the early exercise boundary. We should then carry out a new Monte Carlo simulation using the early exercise boundary to value the option.

Problem 26.22.

Consider the situation in Merton's jump diffusion model where the underlying asset is a non-dividend paying stock. The average frequency of jumps is one per year. The average percentage jump size is 2% and the standard deviation of the logarithm of the percentage jump size is 20%. The stock price is 100, the risk-free rate is 5%, the volatility, σ provided by the diffusion part of the process is 15%, and the time to maturity is six months. Use the DerivaGem Application Builder to calculate an implied volatility when the strike price *is 80, 90, 100, 110, and 120. What does the volatility smile or skew that you obtain imply about the probability distribution of the stock price.*

The price of the option using Merton's model can be calculated using the first 20 terms in the formula in Section 26.1. For strike prices of 80, 90, 100, 110, and 120, the option prices are 22.64, 14.17, 7.67, 3.86, and 2.04 respectively. The implied volatilities are 26.00%, 23.64%, 22.85%, 23.65%, and 25.42%, respectively. The smile is similar to that for foreign currencies in Chapter 18. the probability distribution of the asset price in six months has heavier tails than the lognormal distribution.

Problem 26.23.

A three-year convertible bond with a face value of $100 has been issued by company ABC. It pays a coupon of $5 at the end of each year. It can be converted into ABCs equity at the end of the first year or at the end of the second year. At the end of the first year, it can be exchanged for 3.6 shares immediately after the coupon date. At the end of the second year it can be exchanged for 3.5 shares immediately after the coupon date. The current stock price is $25 and the stock price volatility is 25%. No dividends are paid on the stock. The risk-free interest rate is 5% with continuous compounding. The yield on bonds issued by ABC is 7% with continuous compounding and the recovery rate is 30%.

(a) Use a three-step tree to calculate the value of the bond

(b) How much is the conversion option worth?

(c) What difference does it make to the value of the bond and the value of the conversion option if the bond is callable any time within the first two years for $115?

(d) Explain how your analysis would change if there were a dividend payment of $1 on the equity at the six month, 18-month, and 30-month points. Detailed calculations are not required. Hint: Use equation (22.2) to estimate the default intensity.

In this case $\Delta t = 1$, $\lambda = 0.02/0.7 = 0.02857$, $\sigma = 0.25$, $r = 0.05$, $q = 0$, $u = 1.2023$, $d = 0.8318$, $a = 1.0513$, $p_u = 0.6557$, $p_d = 0.3161$, and the probability of a default is 0.0282. The calculations are shown in Figure M26.3. The values at the nodes include the value of the coupon paid just before the node is reached. The value of the convertible is 105.21. the value if there is no conversion is calculated by working out the present value of the coupons and principal at 7%. It is 94.12. The value of the conversion option is therefore 11.09. Calling at node D makes no difference because the bond will be converted at that node anyway. Calling at node B (just before the coupon payment) does make a difference. It reduces the value of the convertible at node B to $115. The value of the bond at node A is reduced by 2.34. This is a reduction in the value of the conversion option. A dividend payment would affect the way the tree is constructed as described in Chapter 19.

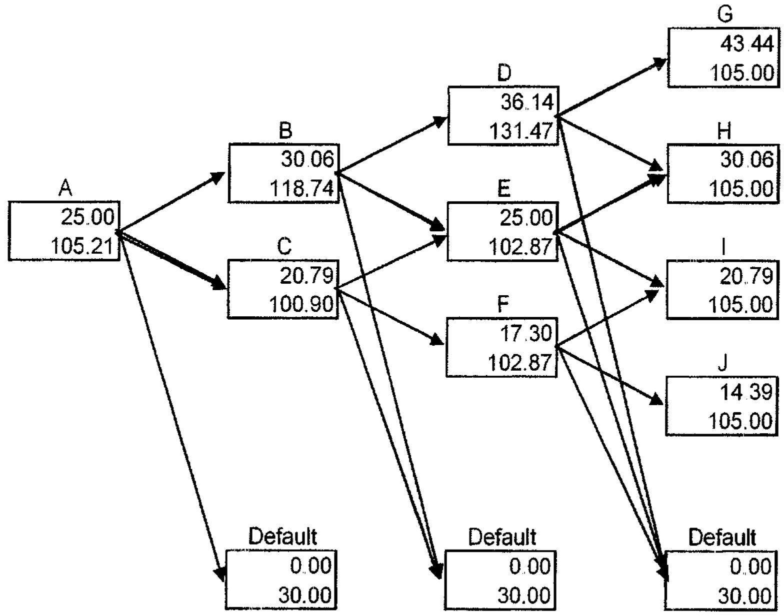


Figure M26.3 Tree for Problem 26.23

CHAPTER 27
Martingales and Measures

Notes for the Instructor

This chapter explains the equivalent martingale measure result. The material in this chapter is important for a full understanding of the way interest rate derivatives are priced in later chapters. A discussion of Black's model (Section 27.6) has been included. This makes the material in Chapter 28 flow more smoothly. For completeness Section 27.9 has been added.

Chapter 27 is more abstract and conceptual than other chapters. Not all instructors will choose to teach the material to undergraduate or even masters-level students.

The first part of the chapter explains the market price of risk. It explains the

$$\mu - r = \lambda\sigma$$

equation for securities dependent on a single variable and

$$\mu - r = \sum \lambda_i \sigma_i$$

for securities dependent on several variables.

The chapter then moves on to discuss martingales and measures. The general approach is to start by deriving results for a world where there is only one source of uncertainty and then point out that the results can be extended to the situation where there are many sources of uncertainty. I start by explaining that, when we use traditional risk-neutral valuation, we are setting the market price of risk to zero, but we do not have to do this. It is sometimes convenient to make other choices for the market price of risk.

A key result is that, if f and g are security prices and there is only one source of uncertainty, f/g is a martingale in a world where the market price of risk is the volatility of g. This is proved in Section 27.3. I refer to a world where the market price of risk is the volatility of g as a world that is "forward risk neutral with respect to g". When g is the money market account we get the traditional risk-neutral world. Other values of the numeraire g lead to other worlds.

A key result, concerned with the impact of a change in the numeraire security g, is proved in Section 27.7. Many instructors who teach this chapter will choose to present this result without proof.

Problem 27.15 is a fairly straightforward application of ideas in the chapter. Problems 27.16 and 27.17 are appropriate for students with good math skills.

QUESTIONS AND PROBLEMS

Problem 27.1.

How is the market price of risk defined for a variable that is not the price of an investment asset?

The market price of risk for a variable that is not the price of a traded security is the market price of risk of a traded security whose price is instantaneously perfectly positively correlated with the variable.

Problem 27.2.

Suppose that the market price of risk for gold is zero. If the storage costs are 1% per annum and the risk-free rate of interest is 6% per annum, what is the expected growth rate in the price of gold? Assume that gold provides no income.

If its market price of risk is zero, gold must, after storage costs have been paid, provide an expected return equal to the risk-free rate of interest. In this case, the expected return after storage costs must be 6% per annum. It follows that the expected growth rate in the price of gold must be 7% per annum.

Problem 27.3.

Consider two securities both of which are dependent on the same market variable. The expected returns from the securities are 8% and 12%. The volatility of the first security is 15%. The instantaneous risk-free rate is 4%. What is the volatility of the second security?

The market price of risk is

$$\frac{\mu - r}{\sigma}$$

This is the same for both securities. From the first security we know it must be

$$\frac{0.08 - 0.04}{0.15} = 0.26667$$

The volatility, σ for the second security is given by

$$\frac{0.12 - 0.04}{\sigma} = 0.26667$$

The volatility is 30%.

Problem 27.4.

An oil company is set up solely for the purpose of exploring for oil in a certain small area of Texas. Its value depends primarily on two stochastic variables: the price of oil and the quantity of proven oil reserves. Discuss whether the market price of risk for the second of these two variables is likely to be positive, negative, or zero.

It can be argued that the market price of risk for the second variable is zero. This is because the risk is unsystematic, i.e., it is totally unrelated to other risks in the economy.

To put this another way, there is no reason why investors should demand a higher return for bearing the risk since the risk can be totally diversified away.

Problem 27.5.

Deduce the differential equation for a derivative dependent on the prices of two non-dividend-paying traded securities by forming a riskless portfolio consisting of the derivative and the two traded securities.

Suppose that the price, f, of the derivative depends on the prices, S_1 and S_2, of two traded securities. Suppose further that:

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dz_1$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dz_2$$

where dz_1 and dz_2 are Wiener processes with correlation ρ. From Ito's lemma [see equation (27A.3)]

$$df = \left(\mu_1 S_1 \frac{\partial f}{\partial S_1} + \mu_2 S_2 \frac{\partial f}{\partial S_2} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} \right.$$
$$\left. + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2}\right) dt + \sigma_1 S_1 \frac{\partial f}{\partial S_1} dz_1 + \sigma_2 S_2 \frac{\partial f}{\partial S_2} dz_2$$

To eliminate the dz_1 and dz_2 we choose a portfolio, Π, consisting of

-1 :	derivative
$+\frac{\partial f}{\partial S_1}$:	first traded security
$+\frac{\partial f}{\partial S_2}$:	second traded security

$$\Pi = -f + \frac{\partial f}{\partial S_1} S_1 + \frac{\partial f}{\partial S_2} S_2$$

$$d\Pi = -df + \frac{\partial f}{\partial S_1} dS_1 + \frac{\partial f}{\partial S_2} dS_2$$

$$= -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2}\right) dt$$

Since the portfolio is instantaneously risk-free it must instantaneously earn the risk-free rate of interest. Hence

$$d\Pi = r\Pi dt$$

Combining the above equations

$$-\left[\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2}\right] dt$$

$$= r\left[-f + \frac{\partial f}{\partial S_1} S_1 + \frac{\partial f}{\partial S_2} S_2\right] dt$$

so that:

$$\frac{\partial f}{\partial t} + rS_1\frac{\partial f}{\partial S_1} + rS_2\frac{\partial f}{\partial S_2} + \frac{1}{2}\sigma_1^2 S_1^2\frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2\frac{\partial^2 f}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2\frac{\partial^2 f}{\partial S_1 \partial S_2} = rf$$

Problem 27.6.

Suppose that an interest rate, x, follows the process

$$dx = a(x_0 - x)\,dt + c\,\sqrt{x}\,dz$$

where a, x_0, and c are positive constants. Suppose further that the market price of risk for x is λ. What is the process for x in the traditional risk-neutral world

The process for x can be written

$$\frac{dx}{x} = \frac{a(x_0 - x)}{x}dt + \frac{c}{\sqrt{x}}dz$$

Hence the expected growth rate in x is:

$$\frac{a(x_0 - x)}{x}$$

and the volatility of x is

$$\frac{c}{\sqrt{x}}$$

In a risk neutral world the expected growth rate should be changed to

$$\frac{a(x_0 - x)}{x} - \lambda\frac{c}{\sqrt{x}}$$

so that the process is

$$\frac{dx}{x} = \left[\frac{a(x_0 - x)}{x} - \lambda\frac{c}{\sqrt{x}}\right]\,dt + \frac{c}{\sqrt{x}}\,dz$$

i.e.

$$dx = \left[a(x_0 - x) - \lambda c\sqrt{x}\right]\,dt + c\sqrt{x}dz$$

Hence the drift rate should be reduced by $\lambda c\sqrt{x}$.

Problem 27.7.

Prove that when the security f provides income at rate q equation (27.9) becomes $\mu + q - r = \lambda\sigma$.(Hint: Form a new security, f^* that provides no income by assuming that all the income from f is reinvested in f.)

As suggested in the hint we form a new security f^* which is the same as f except that all income produced by f is reinvested in f. Assuming we start doing this at time zero, the relationship between f and f^* is

$$f^* = fe^{qt}$$

If μ^* and σ^* are the expected return and volatility of f^*, Ito's lemma shows that

$$\mu^* = \mu + q$$

$$\sigma^* = \sigma$$

From equation (27.9)

$$\mu^* - r = \lambda\sigma^*$$

It follows that

$$\mu + q - r = \lambda\sigma$$

Problem 27.8.

Show that when f and g provide income at rates q_f and q_g respectively, equation (27.15) becomes

$$f_0 = g_0 e^{(q_f - q_g)T} E_g\left(\frac{f_T}{g_T}\right)$$

(Hint: Form new securities f^* and g^* that provide no income by assuming that all the income from f is reinvested in f and all the income in g is reinvested in g.)

As suggested in the hint, we form two new securities f^* and g^* which are the same as f and g at time zero, but are such that income from f is reinvested in f and income from g is reinvested in g. By construction f^* and g^* are non-income producing and their values at time t are related to f and g by

$$f^* = fe^{q_f t} \qquad g^* = ge^{q_g t}$$

From Ito's lemma, the securities g and g^* have the same volatility. We can apply the analysis given in Section 27.3 to f^* and g^* so that from equation (27.15)

$$f_0^* = g_0^* E_g\left(\frac{f_T^*}{g_T^*}\right)$$

or

$$f_0 = g_0 E_g\left(\frac{f_T e^{q_f T}}{g_T e^{q_g T}}\right)$$

or

$$f_0 = g_0 e^{(q_f - q_g)T} E_g\left(\frac{f_T}{g_T}\right)$$

Problem 27.9.

"The expected future value of an interest rate in a risk-neutral world is greater than it is in the real world." What does this statement imply about the market price of risk for (a) an interest rate and (b) a bond price. Do you think the statement is likely to be true? Give reasons.

This statement implies that the interest rate has a negative market price of risk. Since bond prices and interest rates are negatively correlated, the statement implies that the market price of risk for a bond price is positive. The statement is reasonable. When interest rates increase, there is a tendency for the stock market to decrease. This implies that interest rates have negative systematic risk, or equivalently that bond prices have positive systematic risk.

Problem 27.10.

The variable S is an investment asset providing income at rate q measured in currency A. It follows the process

$$dS = \mu_S S\,dt + \sigma_S S\,dz$$

in the real world. Defining new variables as necessary, give the process followed by S, and the corresponding market price of risk, in

(a) A world that is the traditional risk-neutral world for currency A.

(b) A world that is the traditional risk-neutral world for currency B.

(c) A world that is forward risk neutral with respect to a zero-coupon currency A bond maturing at time T.

(d) A world that is forward risk neutral with respect to a zero coupon currency B bond maturing at time T.

(a) In the traditional risk-neutral world the process followed by S is

$$dS = (r - q)S\,dt + \sigma_S S\,dz$$

where r is the instantaneous risk-free rate. The market price of dz-risk is zero.

(b) In the traditional risk-neutral world for currency B the process is

$$dS = (r - q + \rho_{QS}\sigma_S\sigma_Q)S\,dt + \sigma_S S\,dz$$

where Q is the exchange rate (units of A per unit of B), σ_Q is the volatility of Q and ρ_{QS} is the coefficient of correlation between Q and S. The market price of dz-risk is $\rho_{QS}\sigma_Q$

(c) In a world that is forward risk neutral with respect to a zero-coupon bond in currency A maturing at time T

$$dS = (r - q + \sigma_S\sigma_P)S\,dt + \sigma_S S\,dz$$

where σ_P is the bond price volatility. The market price of dz-risk is σ_P

(d) In a world that is forward risk neutral with respect to a zero-coupon bond in currency B maturing at time T

$$dS = (r - q + \sigma_S\sigma_P + \rho_{FS}\sigma_S\sigma_F)S\,dt + \sigma_S S\,dz$$

where F is the forward exchange rate, σ_F is the volatility of F (units of A per unit of B, and ρ_{FS} is the correlation between F and S. The market price of dz-risk is $\sigma_P + \rho_{FS}\sigma_F$.

Problem 27.11.

Explain the difference between the way a forward interest rate is defined and the way the forward values of other variables such as stock prices, commodity prices, and exchange rates are defined.

The forward value of a stock price, commodity price, or exchange rate is the delivery price in a forward contract that causes the value of the forward contract to be zero. A forward bond price is calculated in this way. However, a forward interest rate is the interest rate implied by the forward bond price.

Problem 27.12.

Prove the result in Section 27.5 that when

$$df = \left[r + \sum_{i=1}^{n}\lambda_i\sigma_{f,i}\right] f\,dt + \sum_{i=1}^{n}\sigma_{f,i} f\,dz_i$$

and

$$dg = \left[r + \sum_{i=1}^{n}\lambda_i\sigma_{g,i}\right] g\,dt + \sum_{i=1}^{n}\sigma_{g,i} g\,dz_i$$

with the dz_i uncorrelated, f/g is a martingale for $\lambda_i = \sigma_{g,i}$.

Equation (27A.4) in the appendix to Chapter 27 gives:

$$d\ln f = \left[r + \sum_{i=1}^{n}(\lambda_i\sigma_{f,i} - \sigma_{f,i}^2/2)\right] dt + \sum_{i=1}^{n}\sigma_{f,i}dz_i$$

$$d\ln g = \left[r + \sum_{i=1}^{n}(\lambda_i\sigma_{g,i} - \sigma_{g,i}^2/2)\right] dt + \sum_{i=1}^{n}\sigma_{g,i}dz_i$$

so that

$$d\ln\frac{f}{g} = d(\ln f - \ln g) = \left[\sum_{i=1}^{n}(\lambda_i\sigma_{f,i} - \lambda_i\sigma_{g,i} - \sigma_{f,i}^2/2 + \sigma_{g,i}^2/2)\right] dt + \sum_{i=1}^{n}(\sigma_{f,i} - \sigma_{g,i})dz_i$$

Applying Ito's lemma again

$$d\frac{f}{g} = \frac{f}{g}\left[\sum_{i=1}^{n}(\lambda_i\sigma_{f,i} - \lambda_i\sigma_{g,i} - \sigma_{f,i}^2/2 + \sigma_{g,i}^2/2) + (\sigma_{f,i} - \sigma_{g,i})^2\right] dt + \frac{f}{g}\sum_{i=1}^{n}(\sigma_{f,i} - \sigma_{g,i})dz_i$$

When $\lambda_i = \sigma_{g,i}$ the coefficient of dt is zero and f/g is a martingale.

Problem 27.13.

Prove equation (27.33) in Section 27.7.

Consider the case where v depends on m traded securities f_i, f_2, ..., f_m and the ith component of the volatility of f_j is $\sigma_{i,j}$. With the notation in Section 27.7 when the numeraire changes from g to h the expected growth rate of f_j changes by

$$\sum_{i=1}^{n}(\sigma_{h,i} - \sigma_{g,i})\sigma_{i,j}$$

Equation (27A.5) shows that the drift rate v changes by

$$\sum_{j=1}^{m}\frac{\partial v}{\partial f_j}\sum_{i=1}^{n}(\sigma_{h,i} - \sigma_{g,i})\sigma_{i,j}f_j$$

The ith component of the volatility of v, $\sigma_{v,i}$ is from equation (27A.5) given by

$$v\sigma_{v,i} = \sum_{j=1}^{m}\frac{\partial v}{\partial f_j}\sigma_{i,j}f_j$$

so that the drift rate of v changes by

$$\sum_{i=1}^{n}(\sigma_{h,i} - \sigma_{g,i})v\sigma_{v,i}$$

This is the same as saying that the growth rate of v changes by

$$\sum_{i=1}^{n}(\sigma_{h,i} - \sigma_{g,i})\sigma_{v,i}$$

and proves equation (27.33).

Problem 27.14.

Show that when $w = h/g$ and h and g are each dependent on n Wiener processes, the ith component of the volatility of w is the ith component of the volatility of h minus the ith component of the volatility of g. Use this to prove the result that if σ_U is the volatility *of U and σ_V is the volatility of V then the volatility of U/V is $\sqrt{\sigma_U^2 + \sigma_V^2 - 2\rho\sigma_U\sigma_V}$. (HINT Use the result in footnote 7.)*

Equation (27A.4) in the appendix to Chapter 27 gives:

$$d\ln h = \ldots + \sum_{i=1}^{n} \sigma_{h,i} dz_i$$

$$d\ln g = \ldots + \sum_{i=1}^{n} \sigma_{g,i} dz_i$$

so that

$$d\ln\frac{h}{g} = \ldots + \sum_{i=1}^{n}(\sigma_{h,i} - \sigma_{g,i})dz_i$$

Applying Ito's lemma again

$$d\frac{h}{g} = \ldots + \frac{h}{g}\sum_{i=1}^{n}(\sigma_{h,i} - \sigma_{g,i})dz_i$$

This proves the result.

ASSIGNMENT QUESTIONS

Problem 27.15.

A security's price is positively dependent on two variables: the price of copper and the yen–dollar exchange rate. Suppose that the market price of risk for these variables is 0.5 and 0.1, respectively. If the price of copper were held fixed, the volatility of the security would be 8% per annum; if the yen–dollar exchange rate were held fixed, the volatility of the security would be 12% per annum. The risk-free interest rate is 7% per annum. What is the expected rate of return from the security? If the two variables are uncorrelated with each other, what is the volatility of the security?

Suppose that S is the security price and μ is the expected return from the security. Then:

$$\frac{dS}{S} = \mu dt + \sigma_1 dz_1 + \sigma_2 dz_2$$

where dz_1 and dz_2 are Wiener processes, $\sigma_1 dz_1$ is the component of the risk in the return attributable to the price of copper and $\sigma_2 dz_2$ is the component of the risk in the return attributable to the yen-dollar exchange rate.

If the price of copper is held fixed, $dz_1 = 0$ and:

$$\frac{dS}{S} = \mu dt + \sigma_2 dz_2$$

Hence σ_2 is 8% per annum or 0.08. If the yen-dollar exchange rate is held fixed, $dz_2 = 0$ and:

$$\frac{dS}{S} = \mu dt + \sigma_1 dz_1$$

Hence σ_1 is 12% per annum or 0.12.

From equation (27.13)

$$\mu - r = \lambda_1\sigma_1 + \lambda_2\sigma_2$$

where λ_1 and λ_2 are the market prices of risk for copper and the yen-$ exchange rate. In this case, $r = 0.07$, $\lambda_1 = 0.5$ and $\lambda_2 = 0.1$. Therefore

$$\mu - 0.07 = 0.5 \times 0.12 + 0.1 \times 0.08$$

so that

$$\mu = 0.138$$

i.e., the expected return is 13.8% per annum.

If the two variables affecting S are uncorrelated, we can use the result that the sum of normally distributed variables is normal with variance of the sum equal to the sum of the variances. This leads to:

$$\sigma_1 dz_1 + \sigma_2 dz_2 = \sqrt{\sigma_1^2 + \sigma_2^2} dz_3$$

where dz_3 is a Wiener process. Hence the process for S becomes:

$$\frac{dS}{S} = \mu dt + \sqrt{\sigma_1^2 + \sigma_2^2} dz_3$$

If follows that the volatility of S is $\sqrt{\sigma_1^2 + \sigma_2^2}$ or 14.4% per annum.

Problem 27.16.

Suppose that the price of a zero-coupon bond maturing at time T follows the process

$$dP(t, T) = \mu_P P(t, T)\, dt + \sigma_P P(t, T)\, dz$$

and the price of a derivative dependent on the bond follows the process

$$df = \mu_f f\, dt + \sigma_f f\, dz$$

Assume only one source of uncertainty and that f provides no income.

(a) What is the forward price, F, of f for a contract maturing at time T?

(b) What is the process followed by F in a world that is forward risk neutral with respect to $P(t, T)$?

(c) What is the process followed by F in the traditional risk-neutral world?

(d) What is the process followed by f in a world that is forward risk neutral with respect to a bond maturing at time T^* where $T^* \neq T$? Assume that σ_P^* is the volatility of this bond

(a) The no-arbitrage arguments in Chapter 5 show that

$$F(t) = \frac{f(t)}{P(t,T)}$$

(b) From Ito's lemma:

$$d\ln P = (\mu_P - \sigma_P^2/2)\,dt + \sigma_P\,dz$$

$$d\ln f = (\mu_f - \sigma_f^2/2)\,dt + \sigma_f\,dz$$

Therefore

$$d\ln\frac{f}{P} = d(\ln f - \ln P) = (\mu_f - \sigma_f^2/2 - \mu_P + \sigma_P^2/2)\,dt + (\sigma_f - \sigma_P)\,dz$$

so that

$$d\frac{f}{P} = (\mu_f - \mu_P + \sigma_P^2 - \sigma_f\sigma_P)\frac{f}{P}\,dt + (\sigma_f - \sigma_P)\frac{f}{P}\,dz$$

or

$$dF = (\mu_f - \mu_P + \sigma_P^2 - \sigma_f\sigma_P)F\,dt + (\sigma_f - \sigma_P)F\,dz$$

In a world that is forward risk neutral with respect to $P(t,T)$, F has zero drift. The process for F is

$$dF = (\sigma_f - \sigma_P)F\,dz$$

(c) In the traditional risk-neutral world, $\mu_f = \mu_P = r$ where r is the short-term risk-free rate and

$$dF = (\sigma_P^2 - \sigma_f\sigma_P)F\,dt + (\sigma_f - \sigma_P)F\,dz$$

Note that the answers to parts (b) and (c) are consistent with the market price of risk being zero in (c) and σ_P in (b). When the market price of risk is σ_P, $\mu_f = r + \sigma_f\sigma_P$ and $\mu_P = r + \sigma_P^2$.

(d) In a world that is forward risk-neutral with respect to a bond maturing at time T^*, $\mu_P = r + \sigma_P^*\sigma_P$ and $\mu_f = r + \sigma_P^*\sigma_f$ so that

$$dF = [\sigma_P^2 - \sigma_f\sigma_P + \sigma_P^*(\sigma_f - \sigma_P)]F\,dt + (\sigma_f - \sigma_P)F\,dz$$

or

$$dF = (\sigma_f - \sigma_P)(\sigma_P^* - \sigma_P)F\,dt + (\sigma_f - \sigma_P)F\,dz$$

Problem 27.17.

Consider a variable that is not an interest rate

(a) In what world is the futures price of the variable a martingale

(b) In what world is the forward price of the variable a martingale

(c) Defining variables as necessary derive an expression for the difference between the drift of the futures price and the drift of the forward price in the traditional risk-neutral world

(d) Show that your result is consistent with the points made in Section 5.8 about the circumstances when the futures price is above the forward price.

(a) The futures price is a martingale in the traditional risk-neutral world

(b) The forward price for a contract maturing at time T is a martingale is a world that is forward risk neutral with respect to $P(t, T)$

(c) Define σ_P as the volatility of $P(t, T)$ and σ_F as the volatility of the forward price. The forward rate has zero drift in a world that is forward risk neutral with respect to $P(t, T)$. When we move from the traditional world to a world that is forward risk neutral with respect to $P(t, T)$ the volatility of the numeraire ratio is σ_P and the drift increases by $\rho_{PF}\sigma_P\sigma_F$ where ρ_{PF} is the correlation between $P(t, T)$ and the forward price. It follows that the drift of the forward price in the traditional risk neutral world is $-\rho_{PF}\sigma_P\sigma_F$. The drift of the futures price is zero in the traditional risk neutral world. It follows that the excess of the drift of the futures price over the forward price is $\rho_{PF}\sigma_P\sigma_F$.

(d) P is inversely correlated with interest rates. It follows that when the correlation between interest rates and F is positive the futures price has a lower drift than the forward price. The futures and forward prices are the same at maturity. It follows that the futures price is above the forward price prior to maturity. This is consistent with Section 5.8. Similarly when the correlation between interest rates and F is negative the future price is below the forward price prior to maturity.

CHAPTER 28

Interest Rate Derivatives: The Standard Market Models

Notes for the Instructor

This chapter explains how the market uses Black's model (which was originally developed for options on commodity futures) can be used for three of the most popular over-the-counter interest rate options: European bond options, caps/floors, and European swap options. The implied volatilities quoted for these products (see, for example, Tables 28.1 and 28.2) are the volatilities implied by Black's model. Day count conventions are important in interest rate derivatives. In general, I find it best to initially ignore day count conventions when products are being explained (or, to be more precise, to assume actual/actual and that days are divisible). Later when students are comfortable with a product, the impact of day count conventions can be explained. The chapter follows this approach. There are sections on day count conventions after products have been explained.

The chapter uses the material in Chapter 27 to show that each application of Black's model is internally consistent. These parts of the chapter can be skipped if Chapter 27 has not been covered.

I spend some time discussing each of the three products (European bond options, caps/floors, and European swap options). In addition to the pricing formulas I explain a number of market conventions. For example, (a) the yield volatility quoted for bond options is usually converted to a price volatility, using the duration relationship, before Black's model is used; (b) there is no payoff from a cap on the first reset date; (c) a 3×5 swap option is a three year option to enter into a swap which will last a further five years.

Problems 28.21 to 28.25 are fairly straightforward assignment questions. 28.23, 28.24, and 28.25 require the use of DerivaGem.

QUESTIONS AND PROBLEMS

Problem 28.1.

A company caps three-month LIBOR at 10% per annum. The principal amount is $20 million. On a reset date, three-month LIBOR is 12% per annum. What payment would this lead to under the cap? When would the payment be made?

An amount

$$\$20,000,000 \times 0.02 \times 0.25 = \$100,000$$

would be paid out 3 months later.

Problem 28.2.

Explain why a swap option can be regarded as a type of bond option.

A swap option (or swaption) is an option to enter into an interest rate swap at a certain time in the future with a certain fixed rate being used. An interest rate swap can be regarded as the exchange of a fixed-rate bond for a floating-rate bond. A swaption is therefore the option to exchange a fixed-rate bond for a floating-rate bond. The floating-rate bond will be worth its face value at the beginning of the life of the swap. The swaption is therefore an option on a fixed-rate bond with the strike price equal to the face value of the bond.

Problem 28.3.

Use the Black's model to value a one-year European put option on a 10-year bond. Assume that the current value of the bond is \$125, the strike price is \$110, the one-year interest rate is 10% per annum, the bond's forward price volatility is 8% per annum, and the present value of the coupons to be paid during the life of the option is \$10.

In this case, $F_0 = (125 - 10)e^{0.1\times 1} = 127.09$, $K = 110$, $P(0,T) = e^{-0.1\times 1}$, $\sigma_B = 0.08$, and $T = 1.0$.

$$d_1 = \frac{\ln(127.09/110) + (0.08^2/2)}{0.08} = 1.8456$$
$$d_2 = d_1 - 0.08 = 1.7656$$

From equation (28.2) the value of the put option is

$$110e^{-0.1\times 1}N(-1.7656) - 127.09e^{-0.1\times 1}N(-1.8456) = 0.12$$

or \$0.12.

Problem 28.4.

Explain carefully how you would use (a) spot volatilities and (b) flat volatilities to value a five-year cap.

When spot volatilities are used to value a cap, a different volatility is used to value each caplet. When flat volatilities are used, the same volatility is used to value each caplet within a given cap. Spot volatilities are a function of the maturity of the caplet. Flat volatilities are a function of the maturity of the cap.

Problem 28.5.

Calculate the price of an option that caps the three-month rate, starting in 15 months time, at 13% (quoted with quarterly compounding) on a principal amount of \$1,000. The forward interest rate for the period in question is 12% per annum (quoted with quarterly compounding), the 18-month risk-free interest rate (continuously compounded) is 11.5% per annum, and the volatility of the forward rate is 12% per annum.

In this case $L = 1000$, $\delta_k = 0.25$, $F_k = 0.12$, $R_K = 0.13$, $r = 0.115$, $\sigma_k = 0.12$, $t_k = 1.25$, $P(0, t_{k+1}) = 0.8416$.

$$L\delta_k = 250$$

$$d_1 = \frac{\ln(0.12/0.13) + 0.12^2 \times 1.25/2}{0.12\sqrt{1.25}} = -0.5295$$
$$d_2 = -0.5295 - 0.12\sqrt{1.25} = -0.6637$$

The value of the option is

$$250 \times 0.8416 \times [0.12N(-0.5295) - 0.13N(-0.6637)]$$
$$= 0.59$$

or \$0.59.

Problem 28.6.

A bank uses Black's model to price European bond options. Suppose that an implied price volatility for a 5-year option on a bond maturing in 10 years is used to price a 9-year option on the bond. Would you expect the resultant price to be too high or too low? Explain.

The implied volatility measures the standard deviation of the logarithm of the bond price at the maturity of the option divided by the square root of the time to maturity. In the case of a five year option on a ten year bond, the bond has five years left at option maturity. In the case of a nine year option on a ten year bond it has one year left. The standard deviation of a one year bond price observed in nine years can be normally be expected to be considerably less than that of a five year bond price observed in five years. (See Figure 28.1.) We would therefore expect the price to be too high.

Problem 28.7.

Calculate the value of a four-year European call option on bond that will mature five years from today using Black's model. The five-year cash bond price is \$105, the cash price of a four-year bond with the same coupon is \$102, the strike price is \$100, the four-year risk-free interest rate is 10% per annum with continuous compounding, and the volatility for the bond price in four years is 2% per annum.

The present value of the principal in the four year bond is $100e^{-4\times0.1} = 67.032$. The present value of the coupons is, therefore, $102 - 67.032 = 34.968$. This means that the forward price of the five-year bond is

$$(105 - 34.968)e^{4\times0.1} = 104.475$$

The parameters in Black's model are therefore $F_0 = 104.475$, $K = 100$, $r = 0.1$, $T = 4$, and $\sigma = 0.02$.

$$d_1 = \frac{\ln 1.04475 + 0.5 \times 0.02^2 \times 4}{0.02\sqrt{4}} = 1.1144$$
$$d_2 = d_1 - 0.02\sqrt{4} = 1.0744$$

The price of the European call is

$$e^{-0.1\times4}[104.475N(1.1144) - 100N(1.0744)] = 3.19$$

or \$3.19.

Problem 28.8.

If the yield volatility for a five-year put option on a bond maturing in 10 years time is specified as 22%, how should the option be valued? Assume that, based on today's interest rates the modified duration of the bond at the maturity of the option will be 4.2 years and the forward yield on the bond is 7%.

The option should be valued using Black's model in equation (28.2) with the bond price volatility being

$$4.2 \times 0.07 \times 0.22 = 0.0647$$

or 6.47%.

Problem 28.9.

What other instrument is the same as a five-year zero-cost collar where the strike price of the cap equals the strike price of the floor? What does the common strike price equal?

A 5-year zero-cost collar where the strike price of the cap equals the strike price of the floor is the same as an interest rate swap agreement to receive floating and pay a fixed rate equal to the strike price. The common strike price is the swap rate. Note that the swap is actually a forward swap that excludes the first exchange. (See Business Snapshot 28.1)

Problem 28.10.

Derive a put–call parity relationship for European bond options.

There are two way of expressing the put–call parity relationship for bond options. The first is in terms of bond prices:

$$c + I + Ke^{-RT} = p + B$$

where c is the price of a European call option, p is the price of the corresponding European put option, I is the present value of the bond coupon payments during the life of the option, K is the strike price, T is the time to maturity, B is the bond price, and R is the risk-free interest rate for a maturity equal to the life of the options. To prove this we can consider two portfolios. The first consists of a European put option plus the bond; the second consists of the European call option, and an amount of cash equal to the present value of the coupons plus the present value of the strike price. Both can be seen to be worth the same at the maturity of the options.

The second way of expressing the put–call parity relationship is

$$c + Ke^{-RT} = p + F_0e^{-RT}$$

where F_0 is the forward bond price. This can also be proved by considering two portfolios. The first consists of a European put option plus a forward contract on the bond plus the present value of the forward price; the second consists of a European call option plus the present value of the strike price. Both can be seen to be worth the same at the maturity of the options.

Problem 28.11.

Derive a put–call parity relationship for European swap options.

The put–call parity relationship for European swap options is

$$c + V = p$$

where c is the value of a call option to pay a fixed rate of s_K and receive floating, p is the value of a put option to receive a fixed rate of s_K and pay floating, and V is the value of the forward swap underlying the swap option where s_K is received and floating is paid. This can be proved by considering two portfolios. The first consists of the put option; the second consists of the call option and the swap. Suppose that the actual swap rate at the maturity of the options is greater than s_K. The call will be exercised and the put will not be exercised. Both portfolios are then worth zero. Suppose next that the actual swap rate at the maturity of the options is less than s_K. The put option is exercised and the call option is not exercised. Both portfolios are equivalent to a swap where s_K is received and floating is paid. In all states of the world the two portfolios are worth the same at time T. They must therefore be worth the same today. This proves the result.

Problem 28.12.

Explain why there is an arbitrage opportunity if the implied Black (flat) volatility of a cap is different from that of a floor. Do the broker quotes in Table 28.1 present an arbitrage opportunity?

Suppose that the cap and floor have the same strike price and the same time to maturity. The following put–call parity relationship must hold:

$$\text{cap} + \text{swap} = \text{floor}$$

where the swap is an agreement to receive the cap rate and pay floating over the whole life of the cap/floor. If the implied Black volatilities for the cap equals that for the floor, the Black formulas show that this relationship holds. In other circumstances it does not hold and there is an arbitrage opportunity. The broker quotes in Table 28.1 do not present an arbitrage opportunity because the cap offer is always higher than the floor bid and the floor offer is always higher than the cap bid.

Problem 28.13.

When a bond's price is lognormal can the bond's yield be negative? Explain your answer.

Yes. If a zero-coupon bond price at some future time is lognormal, there is some chance that the price will be above par. This in turn implies that the yield to maturity on the bond is negative.

Problem 28.14.

What is the value of a European swap option that gives the holder the right to enter into a 3-year annual-pay swap in four years where a fixed rate of 5% is paid and LIBOR is received? The swap principal is \$10 million. Assume that the yield curve is flat at 5% per annum with annual compounding and the volatility of the swap rate is 20%. Compare your answer to that given by DerivaGem.

In equation (28.10), $L = 10,000,000$, $s_K = 0.05$, $s_0 = 0.05$, $d_1 = 0.2\sqrt{4}/2 = 0.2$, $d_2 = -.2$, and

$$A = \frac{1}{1.05^5} + \frac{1}{1.05^6} + \frac{1}{1.05^7} = 2.2404$$

The value of the swap option (in millions of dollars) is

$$10 \times 2.2404[0.05N(0.2) - 0.05N(-0.2)] = 0.178$$

This is the same as the answer given by DerivaGem. (For the purposes of using the DerivaGem software note that the interest rate is 4.879% with continuous compounding for all maturities.)

Problem 28.15.

Suppose that the yield, R, on a zero-coupon bond follows the process

$$dR = \mu\, dt + \sigma\, dz$$

where μ and σ are functions of R and t, and dz is a Wiener process. Use Ito's lemma to show that the volatility of the zero-coupon bond price declines to zero as it approaches maturity.

The price of the bond at time t is $e^{-R(T-t)}$ where T is the time when the bond matures. Using Itô's lemma the volatility of the bond price is

$$\sigma \frac{\partial}{\partial R} e^{-R(T-t)} = -\sigma(T-t)e^{-R(T-t)}$$

This tends to zero as t approaches T.

Problem 28.16.

Carry out a manual calculation to verify the option prices in Example 28.2.

The cash price of the bond is

$$4e^{-0.05\times 0.50} + 4e^{-0.05\times 1.00} + \ldots + 4e^{-0.05\times 10} + 100e^{-0.05\times 10} = 122.82$$

As there is no accrued interest this is also the quoted price of the bond. The interest paid during the life of the option has a present value of

$$4e^{-0.05\times 0.5} + 4e^{-0.05\times 1} + 4e^{-0.05\times 1.5} + 4e^{-0.05\times 2} = 15.04$$

The forward price of the bond is therefore

$$(122.82 - 15.04)e^{0.05\times 2.25} = 120.61$$

The duration of the bond at option maturity is

$$\frac{0.25\times 4e^{-0.05\times 0.25} + \ldots + 7.75\times 4e^{-0.05\times 7.75} + 7.75\times 100e^{-0.05\times 7.75}}{4e^{-0.05\times 0.25} + 4e^{-0.05\times 0.75} + \ldots + 4e^{-0.05\times 7.75} + 100e^{-0.05\times 7.75}}$$

or 5.99. The bond price volatility is therefore $5.99\times 0.05\times 0.2 = 0.0599$. We can therefore value the bond option using Black's model with $F_0 = 120.61$, $P(0, 2.25) = e^{-0.05\times 2.25} = 0.8936$, $\sigma = 5.99\%$, and $T = 2.25$. When the strike price is the cash price $K = 115$ and the value of the option is 1.78. When the strike price is the quoted price $K = 117$ and the value of the option is 2.41.

Problem 28.17.

Suppose that the 1-year, 2-year, 3-year, 4-year and 5-year zero rates are 6%, 6.4%, 6.7%, 6.9%, and 7%. The price of a 5-year semiannual cap with a principal of $100 at a cap rate of 8% is $3. Use DerivaGem to determine

a. *The 5-year flat volatility for caps and floors*
b. *The floor rate in a zero-cost 5-year collar when the cap rate is 8%*

We choose the Caps and Swap Options worksheet of DerivaGem and choose Cap/Floor as the Underlying Type. We enter the 1-, 2-, 3-, 4-, 5-year zero rates as 6%, 6.4%, 6.7%, 6.9%, and 7.0% in the Term Structure table. We enter Semiannual for the Settlement Frequency, 100 for the Principal, 0 for the Start (Years), 5 for the End (Years), 8% for the Cap/Floor Rate, and $3 for the Price. We select Black-European as the Pricing Model and choose the Cap button. We check the Imply Volatility box and Calculate. The implied volatility is 24.79%. We then uncheck Implied Volatility, select Floor, check Imply Breakeven Rate. The floor rate that is calculated is 6.71%. This is the floor rate for which the floor is worth $3. A collar when the floor rate is 6.71% and the cap rate is 8% has zero cost.

Problem 28.18.

Show that $V_1 + f = V_2$ where V_1 is the value of a swap option to pay a fixed rate of s_K and receive LIBOR between times T_1 and T_2, f is the value of a forward swap to receive a fixed rate of s_K and pay LIBOR between times T_1 and T_2, and V_2 is the value of a swap option to receive a fixed rate of s_K between times T_1 and T_2. Deduce that $V_1 = V_2$ when s_K equals the current forward swap rate.

We prove this result by considering two portfolios. The first consists of the swap option to receive s_K; the second consists of the swap option to pay s_K and the forward swap. Suppose that the actual swap rate at the maturity of the options is greater than s_K. The swap option to pay s_K will be exercised and the swap option to receive s_K will not be exercised. Both portfolios are then worth zero since the swap option to pay s_K is neutralized by the forward swap. Suppose next that the actual swap rate at the maturity of the options is less than s_K. The swap option to receive s_K is exercised and the swap option to pay s_K is not exercised. Both portfolios are then equivalent to a swap where s_K is received and floating is paid. In all states of the world the two portfolios are worth the same at time T_1. They must therefore be worth the same today. This proves the result. When s_K equals the current forward swap rate $f = 0$ and $V_1 = V_2$. A swap option to pay fixed is therefore worth the same as a similar swap option to receive fixed when the fixed rate in the swap option is the forward swap rate.

Problem 28.19.

Suppose that zero rates are as in Problem 28.17. Use DerivaGem to determine the value of an option to pay a fixed rate of 6% and receive LIBOR on a five-year swap starting in one year. Assume that the principal is $100 million, payments are exchanged semiannually, and the swap rate volatility is 21%.

We choose the Caps and Swap Options worksheet of DerivaGem and choose Swap Option as the Underlying Type. We enter 100 as the Principal, 1 as the Start (Years), 6 as the End (Years), 6% as the Swap Rate, and Semiannual as the Settlement Frequency. We choose Black-European as the pricing model, enter 21% as the Volatility and check the Pay Fixed button. We do not check the Imply Breakeven Rate and Imply Volatility boxes. The value of the swap option is 5.63.

Problem 28.20.

Describe how you would (a) calculate cap flat volatilities from cap spot volatilities and (b) calculate cap spot volatilities from cap flat volatilities.

(a) To calculate flat volatilities from spot volatilities we choose a strike rate and use the spot volatilities to calculate caplet prices. We then sum the caplet prices to obtain cap prices and imply flat volatilities from Black's model. The answer is slightly dependent on the strike price chosen. This procedure ignores any volatility smile in cap pricing.

(b) To calculate spot volatilities from flat volatilities the first step is usually to interpolate between the flat volatilities so that we have a flat volatility for each caplet payment date. We choose a strike price and use the flat volatilities to calculate cap prices. By subtracting successive cap prices we obtain caplet prices from which we can imply spot volatilities. The answer is slightly dependent on the strike price chosen. This procedure also ignores any volatility smile in caplet pricing.

ASSIGNMENT QUESTIONS

Problem 28.21.

Consider an eight-month European put option on a Treasury bond that currently has 14.25 years to maturity. The current cash bond price is $910, the exercise price is $900, and the volatility for the bond price is 10% per annum. A coupon of $35 will be paid by the bond in three months. The risk-free interest rate is 8% for all maturities up to one year. Use Black's model to determine the price of the option. Consider both the case where the *strike price corresponds to the cash price of the bond and the case where it corresponds to the quoted price.*

The present value of the coupon payment is

$$35e^{-0.08\times 0.25} = 34.31$$

Equation (28.2) can therefore be used with $F_B = (910-34.31)e^{0.08\times 8/12} = 923.66$, $r = 0.08$, $\sigma_B = 0.10$ and $T = 0.6667$. When the strike price is a cash price, $K = 900$ and

$$d_1 = \frac{\ln(923.66/900) + 0.005 \times 0.6667}{0.1\sqrt{0.6667}} = 0.3587$$

$$d_2 = d_1 - 0.1\sqrt{0.6667} = 0.2770$$

The option price is therefore

$$900e^{-0.08\times 0.6667}N(-0.2770) - 875.69N(-0.3587) = 18.34$$

or \$18.34.

When the strike price is a quoted price 5 months of accrued interest must be added to 900 to get the cash strike price. The cash strike price is $900 + 35 \times 0.8333 = 929.17$. In this case

$$d_1 = \frac{\ln(923.66/929.17) + 0.005 \times 0.6667}{0.1\sqrt{0.6667}} = -0.0319$$

$$d_2 = d_1 - 0.1\sqrt{0.6667} = -0.1136$$

and the option price is

$$929.17e^{-0.08\times 0.6667}N(0.1136) - 875.69N(0.0319) = 31.22$$

or \$31.22.

Problem 28.22.

Calculate the price of a cap on the 90-day LIBOR rate in nine months' time when the principal amount is \$1,000. Use Black's model and the following information:

a. *The quoted nine-month Eurodollar futures price = 92. (Ignore differences between futures and forward rates.)*
b. *The interest-rate volatility implied by a nine-month Eurodollar option = 15% per annum.*
c. *The current 12-month interest rate with continuous compounding = 7.5% per annum.*
d. *The cap rate = 8% per annum. (Assume an actual/360 day count.)*

The quoted futures price corresponds to a forward rate is 8% per annum with quarterly compounding and actual/360. The parameters for Black's model are therefore: $F_k = 0.08$, $K = 0.08$, $R = 0.075$, $\sigma_k = 0.15$, $t_k = 0.75$, and $P(0, t_{k+1}) = e^{-0.075\times 1} = 0.9277$

$$d_1 = \frac{0.5 \times 0.15^2 \times 0.75}{0.15\sqrt{0.75}} = 0.0650$$

$$d_2 = -\frac{0.5 \times 0.15^2 \times 0.75}{0.15\sqrt{0.75}} = -0.0650$$

and the call price, c, is given by

$$c = 0.25 \times 1,000 \times 0.9277\,[0.08N(0.0650) - 0.08N(-0.0.650)] = 0.96$$

Problem 28.23.

Suppose that the LIBOR yield curve is flat at 8% with annual compounding. A swaption gives the holder the right to receive 7.6% in a five-year swap starting in four years. Payments are made annually. The volatility of the forward swap rate is 25% per annum and the principal is \$1 million. Use Black's model to price the swaption. Compare your answer to that given by DerivaGem.

The payoff from the swaption is a series of five cash flows equal to $\max[0.076 - s_T,\ 0]$ in million of dollars where s_T is the five-year swap rate in four years. The value of an annuity that provides \$1 per year at the end of years 5, 6, 7, 8, and 9 is

$$\sum_{i=5}^{9} \frac{1}{1.08^i} = 2.9348$$

The value of the swaption in millions of dollars is therefore

$$2.9348[0.076N(-d_2) - 0.08N(-d_1)]$$

where

$$d_1 = \frac{\ln(0.08/0.076) + 0.25^2 \times 4/2}{0.25\sqrt{4}} = 0.3526$$

and

$$d_2 = \frac{\ln(0.08/0.076) - 0.25^2 \times 4/2}{0.25\sqrt{4}} = -0.1474$$

The value of the swaption is

$$2.9348[0.076N(0.1474) - 0.08N(-0.3526)] = 0.039554$$

or \$39,554. This is the same answer as that given by DerivaGem. Note that for the purposes of using DerivaGem the zero rate is 7.696% continuously compounded for all maturities.

Problem 28.24.

Use the DerivaGem software to value a five-year collar that guarantees that the maximum and minimum interest rates on a LIBOR-based loan (with quarterly resets) are 5% and 7% respectively. The LIBOR zero curve (continuously compounded) is currently flat at 6%. Use a flat volatility of 20%. Assume that the principal is \$100.

We use the Caps and Swap Options worksheet of DerivaGem. To set the zero curve as flat at 6% with continuous compounding, we need only enter 6% for one maturity. To value the cap we select Cap/Floor as the Underlying Type, enter Quarterly for the Settlement Frequency, 100 for the Principal, 0 for the Start (Years), 5 for the End (Years), 7% for the Cap/Floor Rate, and 20% for the Volatility. We select Black-European as the Pricing Model and choose the Cap button. We do not check the Imply Breakeven Rate and Imply Volatility boxes. The value of the cap is 1.565. To value the floor we change the Cap/Floor Rate to 5% and select the Floor button rather than the Cap button. The value is 1.072. The collar is a long position in the cap and a short position in the floor. The value of the collar is therefore

$$1.565 - 1.072 = 0.493$$

Problem 28.25.

Use the DerivaGem software to value a European swap option that gives you the right in two years to enter into a 5-year swap in which you pay a fixed rate of 6% and receive floating. Cash flows are exchanged semiannually on the swap. The 1-year, 2-year, 5-year, and 10-year zero-coupon interest rates (continuously compounded) are 5%, 6%, 6.5%, and 7%, respectively. Assume a principal of $100 and a volatility of 15% per annum. Give an example of how the swap option might be used by a corporation. What bond option is equivalent to the swap option?

We choose the third worksheet of DerivaGem and choose Swap Option as the Underlying Type. We enter 100 as the Principal, 2 as the Start (Years), 7 as the End (Years), 6% as the Swap Rate, and Semiannual as the Settlement Frequency. We also enter the zero curve information. We choose Black-European as the pricing model, enter 15% as the Volatility and check the Pay Fixed button. We do not check the Imply Breakeven Rate and Imply Volatility boxes. The value of the swap option is 4.606. For a company that expects to borrow at LIBOR plus 50 basis points in two years and then enter into a swap to convert to five-year fixed-rate borrowings, the swap guarantees that its effective fixed rate will not be more than 6.5%. The swap option is the same as an option to sell a five-year 6% coupon bond for par in two years.

CHAPTER 29

Convexity, Timing, and Quanto Adjustments

Notes for the Instructor

This chapter uses the results in Chapter 27. It starts by pointing out that when the expected value of a bond price is the forward price the expected value of a bond yield is not the forward yield. To value a product that provides a payoff at time T dependent on a bond yield observed at that time we want in work in a world where the bond price equals its forward price. (This is a world that is forward risk neutral with respect to a zero-coupon bond maturing at time T.) We must therefore make an adjustment to the forward yield. This is referred to as a convexity adjustment. I like to go through Examples 29.1 and 29.2 fairly carefully to make sure students understand what is going on.

Timing adjustments and quanto adjustments are applications of the change of numeraire argument in Section 27.8. Again, when teaching this material, I make heavy use of the examples in the text (29.3, 29.4, and 29.5). I find Siegel's paradox works well as an application of the ideas in Section 29.3.

Problems 29.11, 29.12, and 29.13 are fairly straightforward applications of the material in the chapter. Problem 29.10 is a little more difficult. It can work well as an assignment or for class discussion

QUESTIONS AND PROBLEMS

Problem 29.1.

Explain how you would value a derivative that pays off $100R$ in five years where R is the one-year interest rate (annually compounded) observed in four years. What difference would it make if the payoff were in four years? What difference would it make if the payoff were in six years?

The value of the derivative is $100R_{4,5}P(0,5)$ where $P(0,t)$ is the value of a t-year zero-coupon bond today and R_{t_1,t_2} is the forward rate for the period between t_1 and t_2, expressed with annual compounding. If the payoff is made in four years the value is $100(R_{4,5}+c)P(0,4)$ where c is the convexity adjustment given by equation (29.2). the formula for the convexity adjustment is:

$$c = \frac{4R_{4,5}^2\sigma_{4,5}^2}{(1+R_{4,5})}$$

where σ_{t_1,t_2} is the volatility of the forward rate between times t_1 and t_2.

The expression $100(R_{4,5}+c)$ is the expected payoff in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time four years. If the payoff is made in six years, the value is from equation (29.4) given by

$$100(R_{4,5}+c)P(0,6)\exp\left[-\frac{4\rho\sigma_{4,5}\sigma_{4,6}R_{4,6}\times 2}{1+R_{4,6}}\right]$$

where ρ is the correlation between the (4,5) and (4,6) forward rates. As an approximation we can assume that $\rho = 1$, $\sigma_{4,5} = \sigma_{4,6}$, and $R_{4,5} = R_{4,6}$. Approximating the exponential function we then get the value of the derivative as $100(R_{4,5}-c)P(0,6)$.

Problem 29.2.

Explain whether any convexity or timing adjustments are necessary when

a. *We wish to value a spread option that pays off every quarter the excess (if any) of the five-year swap rate over the three-month LIBOR rate applied to a principal of \$100. The payoff occurs 90 days after the rates are observed.*
b. *We wish to value a derivative that pays off every quarter the three-month LIBOR rate minus the three-month Treasury bill rate. The payoff occurs 90 days after the rates are observed.*

(a) A convexity adjustment is necessary for the swap rate
(b) No convexity or timing adjustments are necessary.

Problem 29.3.

Suppose that in Example 28.3 of Section 28.2 the payoff occurs after one year (i.e., when the interest rate is observed) rather than in 15 months. What difference does this make to the inputs to Black's models?

There are two differences. The discounting is done over a 1.0-year period instead of over a 1.25-year period. Also a convexity adjustment to the forward rate is necessary. From equation (29.2) the convexity adjustment is:

$$\frac{0.07^2\times 0.2^2\times 0.25\times 1}{1+0.25\times 0.07} = 0.00005$$

or about half a basis point.

In the formula for the caplet we set $F_k = 0.07005$ instead of 0.07. This means that $d_1 = -0.5642$ and $d_2 = -0.7642$. With continuous compounding the 15-month rate is 6.5% and the forward rate between 12 and 15 months is 6.94%. The 12 month rate is therefore 6.39% The caplet price becomes

$$0.25\times 10{,}000e^{-0.069394\times 1.0}[0.07005N(-0.5642)-0.08N(-0.7642)] = 5.29$$

or \$5.29.

Problem 29.4.

The yield curve is flat at 10% per annum with annual compounding. Calculate the value of an instrument where, in five years' time, the two-year swap rate (with annual compounding) is received and a fixed rate of 10% is paid. Both are applied to a notional principal of $100. Assume that the volatility of the swap rate is 20% per annum. Explain why the value of the instrument is different from zero.

The convexity adjustment discussed in Section 29.1 leads to the instrument being worth an amount slightly different from zero. Define $G(y)$ as the value as seen in five years of a two-year bond with a coupon of 10% as a function of its yield.

$$G(y) = \frac{0.1}{1+y} + \frac{1.1}{(1+y)^2}$$

$$G'(y) = -\frac{0.1}{(1+y)^2} - \frac{2.2}{(1+y)^3}$$

$$G''(y) = \frac{0.2}{(1+y)^3} + \frac{6.6}{(1+y)^4}$$

It follows that $G'(0.1) = -1.7355$ and $G''(0.1) = 4.6582$ and the convexity adjustment that must be made for the two-year swap- rate is

$$0.5 \times 0.1^2 \times 0.2^2 \times 5 \times \frac{4.6582}{1.7355} = 0.00268$$

We can therefore value the instrument on the assumption that the swap rate will be 10.268% in five years. The value of the instrument is

$$\frac{0.268}{1.1^5} = 0.167$$

or $0.167.

Problem 29.5.

What difference does it make in Problem 29.4 if the swap rate is observed in five years, but the exchange of payments takes place in (a) six years, and (b) seven years? Assume that the volatilities of all forward rates are 20%. Assume also that the forward swap rate for the period between years five and seven has a correlation of 0.8 with the forward interest rate between years five and six and a correlation of 0.95 with the forward interest rate between years five and seven.

In this case we have to make a timing adjustment as well as a convexity adjustment to the forward swap rate. For (a) equation (29.4) shows that the timing adjustment involves multiplying the swap rate by

$$\exp\left[-\frac{0.8 \times 0.20 \times 0.20 \times 0.1 \times 5}{1+0.1}\right] = 0.9856$$

so that it becomes $10.268 \times 0.9856 = 10.120$. The value of the instrument is

$$\frac{0.120}{1.1^6} = 0.068$$

or \$0.068.

For (b) equation (29.4) shows that the timing adjustment involves multiplying the swap rate by

$$\exp\left[-\frac{0.95 \times 0.2 \times 0.2 \times 0.1 \times 2 \times 5}{1+0.1}\right] = 0.9660$$

so that it becomes $10.268 \times 0.966 = 9.919$. The value of the instrument is now

$$-\frac{0.081}{1.1^7} = -0.042$$

or −\$0.042.

Problem 29.6.

The price of a bond at time T, measured in terms of its yield, is $G(y_T)$. Assume geometric Brownian motion for the forward bond yield, y, in a world that is forward risk neutral with respect to a bond maturing at time T. Suppose that the growth rate of the forward bond yield is α and its volatility σ_y.

a. *Use Ito's lemma to calculate the process for the forward bond price in terms of α, σ_y, y, and $G(y)$.*

b. *The forward bond price should follow a martingale in the world considered. Use this fact to calculate an expression for α.*

c. *Show that the expression for α is, to a first approximation, consistent with equation (29.1).*

(a) The process for y is

$$dy = \alpha y\, dt + \sigma_y y\, dz$$

The forward bond price is $G(y)$. From Itô's lemma, its process is

$$d[G(y)] = [G'(y)\alpha y + \frac{1}{2}G''(y)\sigma_y^2 y^2]\, dt + G'(y)\sigma_y y\, dz$$

(b) Since the expected growth rate of $G(y)$ is zero

$$G'(y)\alpha y + \frac{1}{2}G''(y)\sigma_y^2 y^2 = 0$$

or

$$\alpha = -\frac{1}{2}\frac{G''(y)}{G'(y)}\sigma_y^2 y$$

(c) Assuming as an approximation that y always equals its initial value of y_0, this shows that the growth rate of y is

$$-\frac{1}{2}\frac{G''(y_0)}{G'(y_0)}\sigma_y^2 y_0$$

The variable y starts at y_0 and ends as y_T. The convexity adjustment to y_0 when we are calculating the expected value of y_T in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time T is approximately $y_0 T$ times this or

$$-\frac{1}{2}\frac{G''(y_0)}{G'(y_0)}\sigma_y^2 y_0^2 T$$

This is consistent with equation (29.1).

Problem 29.7.

The variable S is an investment asset providing income at rate q measured in currency A. It follows the process

$$dS = \mu_S S\,dt + \sigma_S S\,dz$$

in the real world. Defining new variables as necessary, give the process followed by S, and the corresponding market price of risk, in

(a) A world that is the traditional risk-neutral world for currency A.
(b) A world that is the traditional risk-neutral world for currency B.
(c) A world that is forward risk neutral with respect to a zero-coupon currency A bond maturing at time T.
(d) A world that is forward risk neutral with respect to a zero coupon currency B bond maturing at time T.

(a) In the traditional risk-neutral world the process followed by S is

$$dS = (r - q)S\,dt + \sigma_S S\,dz$$

where r is the instantaneous risk-free rate. The market price of dz-risk is zero.

(b) In the traditional risk-neutral world for currency B the process is

$$dS = (r - q + \rho_{QS}\sigma_S\sigma_Q)S\,dt + \sigma_S S\,dz$$

where Q is the exchange rate (units of A per unit of B), σ_Q is the volatility of Q and ρ_{QS} is the coefficient of correlation between Q and S. The market price of dz-risk is $\rho_{QS}\sigma_Q$

(c) In a world that is forward risk neutral with respect to a zero-coupon bond in currency A maturing at time T

$$dS = (r - q + \sigma_S\sigma_P)S\,dt + \sigma_S S\,dz$$

where σ_P is the bond price volatility. The market price of dz-risk is σ_P

(d) In a world that is forward risk neutral with respect to a zero-coupon bond in currency B maturing at time T

$$dS = (r - q + \sigma_S\sigma_P + \rho_{FS}\sigma_S\sigma_F)S\,dt + \sigma_S S\,dz$$

where F is the forward exchange rate, σ_F is the volatility of F (units of A per unit of B, and ρ_{FS} is the correlation between F and S. The market price of dz-risk is $\sigma_P + \rho_{FS}\sigma_F$.

Problem 29.8.

A call option provides a payoff at time T of $\max(S_T - K, 0)$ yen, where S_T is the dollar price of gold at time T and K is the strike price. Assuming that the storage costs of gold are zero and defining other variables as necessary, calculate the value of the contract.

Define

$P(t,T)$: Price in yen at time t of a bond paying 1 yen at time T
$E_T(\cdot)$: Expectation in world that is forward risk neutral with respect to $P(t,T)$
F: Dollar forward price of gold for a contract maturing at time T
F_0: Value of F at time zero
σ_F: Volatility of F
G: Forward exchange rate (dollars per yen)
σ_G: Volatility of G

We assume that S_T is lognormal. We can work in a world that is forward risk neutral with respect to $P(t,T)$ to get the value of the call as

$$P(0,T)[E_T(S_T)N(d_1) - N(d_2)]$$

where

$$d_1 = \frac{\ln[E_T(S_T)/K] + \sigma_F^2 T/2}{\sigma_F\sqrt{T}}$$

$$d_2 = \frac{\ln[E_T(S_T)/K] - \sigma_F^2 T/2}{\sigma_F\sqrt{T}}$$

The expected gold price in a world that is forward risk-neutral with respect to a zero-coupon dollar bond maturing at time T is F_0. It follows from equation (29.6) that

$$E_T(S_T) = F_0(1 + \rho\sigma_F\sigma_G T)$$

Hence the option price, measured in yen, is

$$P(0,T)[F_0(1 + \rho\sigma_F\sigma_G T)N(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{\ln[F_0(1 + \rho\sigma_F\sigma_G T)/K] + \sigma_F^2 T/2}{\sigma_F\sqrt{T}}$$

$$d_2 = \frac{\ln[F_0(1 + \rho\sigma_F\sigma_G T)/K] - \sigma_F^2 T/2}{\sigma_F\sqrt{T}}$$

Problem 29.9.

Suppose that an index of Canadian stocks currently stands at 400. The Canadian dollar is currently worth 0.70 U.S. dollars. The risk-free interest rates in Canada and the U.S. are constant at 6% and 4%, respectively. The dividend yield on the index is 3%. Define Q as the number of Canadian dollars per U.S dollar and S as the value of the index. The volatility of S is 20%, the volatility of Q is 6%, and the correlation between S and Q is 0.4. Use DerivaGem to determine the value of a two year American-style call option on the index if

(a) It pays off in Canadian dollars the amount by which the index exceeds 400.

(b) It pays off in U.S. dollars the amount by which the index exceeds 400.

(a) The value of the option can be calculated by setting $S_0 = 400$, $K = 400$, $r = 0.06$, $q = 0.03$, $\sigma = 0.2$, and $T = 2$. With 100 time steps the value (in Canadian dollars) is 52.92.

(b) The growth rate of the index using the CDN numeraire is $0.06 - 0.03$ or 3%. When we switch to the USD numeraire we increase the growth rate of the index by $0.4 \times 0.2 \times 0.06$ or 0.48% per year to 3.48%. The option can therefore be calculated using DerivaGem with $S_0 = 400$, $K = 400$, $r = 0.04$, $q = 0.04 - 0.0348 = 0.0052$, $\sigma = 0.2$, and $T = 2$. With 100 time steps DerivaGem gives the value as 57.51.

ASSIGNMENT QUESTIONS

Problem 29.10.

Consider an instrument that will pay off S dollars in two years where S is the value of the Nikkei index. The index is currently 20,000. The dollar–yen exchange rate (yen per dollar) is 100. The correlation between the exchange rate and the index is 0.3 and the dividend yield on the index is 1% per annum. The volatility of the Nikkei index is 20% and the volatility of the yen–dollar exchange rate is 12%. The interest rates (assumed constant) in the U.S. and Japan are 4% and 2%, respectively.

(a) What is the value of the instrument

(b) Suppose that the exchange rate at some point during the life of the instrument is Q and the level of the index is S. Show that a U.S. investor can create a portfolio that changes in value by approximately ΔS dollar when the index changes in value by ΔS yen by investing S dollars in the Nikkei and shorting SQ yen.

(c) Confirm that this is correct by supposing that the index changes from 20,000 to 20,050 and the exchange rate changes from 100 to 99.7.

(d) How would you delta hedge the instrument under consideration?

(a) We require the expected value of the Nikkei index in a dollar risk-neutral world. In a yen risk-neutral world the expected value of the index is $20,000e^{(0.02-0.01)\times 2} = 20,404.03$. In a dollar risk-neutral world the analysis in Section 29.3 shows that this becomes

$$20,404.03e^{0.3\times 0.20\times 0.12\times 2} = 20,699.97$$

The value of the instrument is therefore

$$20,699.97e^{-0.04\times 2} = 19,108.48$$

(b) An amount SQ yen is invested in the Nikkei. Its value in yen changes to

$$SQ\left(1+\frac{\Delta S}{S}\right)$$

In dollars this is worth

$$SQ\frac{1+\Delta S/S}{Q+\Delta Q}$$

where ΔQ is the increase in Q. When terms of order two and higher are ignored, the dollar value becomes

$$S(1+\Delta S/S-\Delta Q/Q)$$

The gain on the Nikkei position is therefore $\Delta S - S\Delta Q/Q$
When SQ yen are shorted the gain in dollars is

$$SQ\left(\frac{1}{Q}-\frac{1}{Q+\Delta Q}\right)$$

This equals $S\Delta Q/Q$ when terms of order two and higher are ignored. The gain on the whole position is therefore ΔS as required.

(c) In this case the investor invests \$20,000 in the Nikkei. The investor converts the funds to yen and buys 100 times the index. The index rises to 20,050 so that the investment becomes worth 2,005,000 yen or

$$\frac{2,005,000}{99.7} = 20,110.33$$

dollars. The investor therefore gains \$110.33. The investor also shorts 2,000,000 yen. The value of the yen changes from \$0.0100 to \$0.01003. The investor therefore loses $0.00003 \times 2,000,000 = 60$ dollars on the short position. The net gain is 50.33 dollars. This is close to the required gain of \$50.

(d) Suppose that the value of the instrument is V. When the index changes by ΔS yen the value of the instrument changes by

$$\frac{\partial V}{\partial S}\Delta S$$

dollars. We can calculate $\partial V/\partial S$. Part (b) of this question shows how to manufacture an instrument that changes by ΔS dollars. This enables us to delta-hedge our exposure to the index.

Problem 29.11.

Suppose that the LIBOR yield curve is flat at 8% (with continuous compounding). The payoff from a derivative occurs in four years. It is equal to the five-year rate minus the two-year rate at this time, applied to a principal of $100 with both rates being continuously compounded. (The payoff can be positive or negative.) Calculate the value of the derivative. Assume that the volatility for all rates is 25%. What difference does it make if the payoff occurs in five years instead of four year? Assume all rates are perfectly correlated.

To calculate the convexity adjustment for the five-year rate define the price of a five year bond, as a function of its yield as

$$G(y) = e^{-5y}$$

$$G'(y) = -5e^{-5y}$$

$$G''(y) = 25e^{-5y}$$

The convexity adjustment is

$$0.5 \times 0.08^2 \times 0.25^2 \times 4 \times 5 = 0.004$$

Similarly for the two year rate the convexity adjustment is

$$0.5 \times 0.08^2 \times 0.25^2 \times 4 \times 2 = 0.0016$$

We can therefore value the derivative by assuming that the five year rate is 8.4% and the two-year rate is 8.16%. The value of the derivative is

$$0.24e^{-0.08\times 4} = 0.174$$

If the payoff occurs in five years rather than four years it is necessary to make a timing adjustment. From equation (29.4) this involves multiplying the forward rate by

$$\exp\left[-\frac{1 \times 0.25 \times 0.25 \times 0.08 \times 4 \times 1}{1.08}\right] = 0.98165$$

The value of the derivative is

$$0.24 \times 0.98165e^{-0.08\times 5} = 0.158$$

Problem 29.12.

Suppose that the payoff from a derivative will occur in ten years and will equal the three-year U.S. dollar swap rate for a semiannual-pay swap observed at that time applied to a certain principal. Assume that the yield curve is flat at 8% (semiannually compounded) per annum in dollars and 3% (semiannually compounded) in yen. The forward swap rate *volatility is 18%, the volatility of the ten year "yen per dollar" forward exchange rate is 12%, and the correlation between this exchange rate and U.S. dollar interest rates is 0.25.*

a. *What is the value of the derivative if the swap rate is applied to a principal of \$100 million so that the payoff is in dollars?*

b. *What is its value of the derivative if the swap rate is applied to a principal of 100 million yen so that the payoff is in yen?*

(a) In this case we must make a convexity adjustment to the forward swap rate. Define

$$G(y) = \sum_{i=1}^{6} \frac{4}{(1+y/2)^i} + \frac{100}{(1+y/2)^6}$$

so that

$$G'(y) = -\sum_{i=1}^{6} \frac{2i}{(1+y/2)^{i+1}} + \frac{300}{(1+y/2)^7}$$

$$G''(y) = \sum_{i=1}^{6} \frac{i(i+1)}{(1+y/2)^{i+2}} + \frac{1050}{(1+y/2)^8}$$

$G'(0.08) = -262.11$ and $G''(0.08) = 853.29$ so that the convexity adjustment is

$$\frac{1}{2} \times 0.08^2 \times 0.18^2 \times 10 \times \frac{853.29}{262.11} = 0.00338$$

The adjusted forward swap rate is $0.08 + 0.00338 = 0.08338$ and the value of the derivative in millions of dollars is

$$\frac{0.08338 \times 100}{1.04^{20}} = 3.805$$

(b) When the swap rate is applied to a yen principal we must make a quanto adjustment in addition to the convexity adjustment. From Section 29.3 this involves multiplying the forward swap rate by $e^{-0.25\times0.12\times0.18\times10} = 0.9474$. (Note that the correlation is the correlation between the dollar per yen exchange rate and the swap rate. It is therefore -0.25 rather than $+0.25$.) The value of the derivative in millions of yen is

$$\frac{0.08338 \times 0.9474 \times 100}{1.015^{20}} = 5.865$$

Problem 29.13.

The payoff from a derivative will occur in 8 years. It will equal the average of the one-year interest rates observed at times 5, 6, 7, and 8 years applied to a principal of \$1,000. The yield curve is flat at 6% with annual compounding and the volatilities of all rates are 16%. Assume perfect correlation between all rates. What is the value of the derivative?

No adjustment is necessary for the forward rate applying to the period between years seven and eight. Using this, we can deduce from equation (29.4) that the forward rate applying to the period between years five and six must be multiplied by

$$\exp\left[-\frac{1 \times 0.16 \times 0.16 \times 0.06 \times 5 \times 2}{1.06}\right] = 0.9856$$

Similarly the forward rate applying to the period between year six and year seven must be multiplied by

$$\exp\left[-\frac{1 \times 0.16 \times 0.16 \times 0.06 \times 6 \times 1}{1.06}\right] = 0.9913$$

Similarly the forward rate applying to the period between year eight and nine must be multiplied by

$$\exp\left[\frac{1 \times 0.16 \times 0.16 \times 0.06 \times 8 \times 1}{1.06}\right] = 1.0117$$

The adjusted forward average interest rate is therefore

$$0.25 \times (0.08 \times 0.9856 + 0.08 \times 0.9913 + 0.08 + 0.08 \times 1.0117) = 0.07977$$

The value of the derivative is

$$0.7977 \times 1000 \times 1.06^{-8} = 50.05$$

CHAPTER 30

Interest Rate Derivatives: Models of the Short Rate

Notes for the Instructor

This chapter covers equilibrium and no-arbitrage models of the short rate. The first part of the chapter explains that if we know all the details of the behavior of the short-term interest rate in a risk-neutral world, we can deduce the behavior of the whole term structure. It reviews the well known equilibrium models such as Vasicek and CIR.

The second part of the chapter on no-arbitrage models explains how one-factor models of the short rate can be constructed so that they are exactly consistent with the zero rates observed in the market today. These types of models are popular with practitioners. The chapter describes a trinomial tree-building procedures that can be used to represent a wide range of different models. DerivaGem can be used to display trees in class as this material is taught.

Problems 30.22 to 30.26 can be used as assignment questions. All make use of DerivaGem.

QUESTIONS AND PROBLEMS

Problem 30.1.

What is the difference between an equilibrium model and a no-arbitrage model?

Equilibrium models usually start with assumptions about economic variables and derive the behavior of interest rates. The initial term structure is an output from the model. In a no-arbitrage model the initial term structure is an input. The behavior of interest rates in a no-arbitrage model is designed to be consistent with the initial term structure.

Problem 30.2.

Suppose that the short rate is currently 4% and its standard deviation is 1% per annum. What happens to the standard deviation when the short rate increases to 8% in (a) Vasicek's model; (b) Rendleman and Bartter's model; and (c) the Cox, Ingersoll, and Ross model?

In Vasicek's model the standard deviation stays at 1%. In the Rendleman and Bartter model the standard deviation is proportional to the level of the short rate. When the short rate increases from 4% to 8% the standard deviation increases from 1% to 2%. In the Cox, Ingersoll, and Ross model the standard deviation of the short rate is proportional to the square root of the short rate. When the short rate increases from 4% to 8% the standard deviation of the short rate increases from 1% to 1.414%.

Problem 30.3.

If a stock price were mean reverting or followed a path-dependent process there would be market inefficiency. Why is there not a market inefficiency when the short-term interest rate does so?

If the price of a traded security followed a mean-reverting or path-dependent process there would be a market inefficiency. The short-term interest rate is not the price of a traded security. In other words we cannot trade something whose price is always the short-term interest rate. There is therefore no market inefficiency when the short-term interest rate follows a mean-reverting or path-dependent process. We can trade bonds and other instruments whose prices do depend on the short rate. The prices of these instruments do not follow mean-reverting or path-dependent processes.

Problem 30.4.

Explain the difference between a one-factor and a two-factor interest rate model.

In a one-factor model there is one source of uncertainty driving all rates. This usually means that in any short period of time all rates move in the same direction (but not necessarily by the same amount). In a two-factor model, there are two sources of uncertainty driving all rates. The first source of uncertainty usually gives rise to a roughly parallel shift in rates. The second gives rise to a twist where long and short rates moves in opposite directions.

Problem 30.5.

Can the approach described in Section 30.4 for decomposing an option on a coupon-bearing bond into a portfolio of options on zero-coupon bonds be used in conjunction with a two-factor model? Explain your answer.

No. The approach in Section 30.4 relies on the argument that, at any given time, all bond prices are moving in the same direction. This is not true when there is more than one factor.

Problem 30.6.

Suppose that $a = 0.1$ and $b = 0.1$ in both the Vasicek and the Cox, Ingersoll, Ross model. In both models, the initial short rate is 10% and the initial standard deviation of the short rate change in a short time Δt is $0.02\sqrt{\Delta t}$. Compare the prices given by the models for a zero-coupon bond that matures in year 10.

In Vasicek's model, $a = 0.1$, $b = 0.1$, and $\sigma = 0.02$ so that

$$B(t, t+10) = \frac{1}{0.1}(1 - e^{-0.1\times 10}) = 6.32121$$

$$A(t, t+10) = \exp\left[\frac{(6.32121 - 10)(0.1^2 \times 0.1 - 0.0002)}{0.01} - \frac{0.0004 \times 6.32121^2}{0.4}\right]$$

$$= 0.71587$$

The bond price is therefore $0.71587e^{-6.32121\times0.1} = 0.38046$

In the Cox, Ingersoll, and Ross model, $a = 0.1$, $b = 0.1$ and $\sigma = 0.02/\sqrt{0.1} = 0.0632$. Also

$$\gamma = \sqrt{a^2 + 2\sigma^2} = 0.13416$$

Define

$$\beta = (\gamma + a)(e^{10\gamma} - 1) + 2\gamma = 0.92992$$

$$B(t, t+10) = \frac{2(e^{10\gamma} - 1)}{\beta} = 6.07650$$

$$A(t, t+10) = \left(\frac{2\gamma e^{5(a+\gamma)}}{\beta}\right)^{2ab/\sigma^2} = 0.69746$$

The bond price is therefore $0.69746e^{-6.07650\times0.1} = 0.37986$

Problem 30.7.

Suppose that $a = 0.1$, $b = 0.08$, and $\sigma = 0.015$ in Vasicek's model with the initial value of the short rate being 5%. Calculate the price of a one-year European call option on a zero-coupon bond with a principal of \$100 that matures in three years when the strike price is \$87.

Using the notation in the text, $s = 3$, $T = 1$, $L = 100$, $K = 87$, and

$$\sigma_P = \frac{0.015}{0.1}(1 - e^{-2\times0.1})\sqrt{\frac{1 - e^{-2\times0.1\times1}}{2\times0.1}} = 0.025886$$

From equation (30.6), $P(0,1) = 0.94988$, $P(0,3) = 0.85092$, and $h = 1.14277$ so that equation (30.20) gives the call price as call price is

$$100 \times 0.85092 \times N(1.14277) - 87 \times 0.94988 \times N(1.11688) = 2.59$$

or \$2.59.

Problem 30.8.

Repeat Problem 30.7 valuing a European put option with a strike of \$87. What is the put–call parity relationship between the prices of European call and put options? Show that the put and call option prices satisfy put–call parity in this case.

As mentioned in the text, equation (30.20) for a call option is essentially the same as Black's model. There is a typo in the first printing of the book. By analogy with Black's formulas corresponding expression for a put option is

$$KP(0,T)N(-h + \sigma_P) - LP(0,s)N(-h)$$

In this case the put price is

$$87 \times 0.94988 \times N(-1.11688) - 100 \times 0.85092 \times N(-1.14277) = 0.14$$

Since the underlying bond pays no coupon, put–call parity states that the put price plus the bond price should equal the call price plus the present value of the strike price. The bond price is 85.09 and the present value of the strike price is $87 \times 0.94988 = 82.64$. Put–call parity is therefore satisfied:

$$82.64 + 2.59 = 85.09 + 0.14$$

Problem 30.9.

Suppose that $a = 0.05$, $b = 0.08$, and $\sigma = 0.015$ in Vasicek's model with the initial short-term interest rate being 6%. Calculate the price of a 2.1-year European call option on a bond that will mature in three years. Suppose that the bond pays a coupon of 5% semiannually. The principal of the bond is 100 and the strike price of the option is 99. The strike price is the cash price (not the quoted price) that will be paid for the bond.

As explained in Section 30.4, the first stage is to calculate the value of r at time 2.1 years which is such that the value of the bond at that time is 99. Denoting this value of r by r^*, we must solve

$$2.5A(2.1, 2.5)e^{-B(2.1,2.5)r^*} + 102.5A(2.1, 3.0)e^{-B(2.1,3.0)r^*} = 99$$

where the A and B functions are given by equations (30.7) and (30.8). The solution to this is $r^* = 0.066$. Since

$$2.5A(2.1, 2.5)e^{-B(2.1,2.5)\times 0.066} = 2.43473$$

and

$$102.5A(2.1, 3.0)e^{-B(2.1,3.0)\times 0.063} = 96.56438$$

the call option on the coupon-bearing bond can be decomposed into a call option with a strike price of 2.43473 on a bond that pays off 2.5 at time 2.5 years and a call option with a strike price of 96.56438 on a bond that pays off 102.5 at time 3.0 years. Equation (30.20) shows that the value of the first option is 0.009085 and the value of the second option is 0.806143. The total value of the option is therefore 0.815238.

Problem 30.10.

Use the answer to Problem 30.9 and put–call parity arguments to calculate the price of a put option that has the same terms as the call option in Problem 30.9.

Put-call parity shows that:

$$c + I + PV(K) = p + B_0$$

or

$$p = c + PV(K) - (B_0 - I)$$

where c is the call price, K is the strike price, I is the present value of the coupons, and B_0 is the bond price. In this case $c = 0.8152$, $PV(K) = 99 \times P(0,2.1) = 87.1222$, $B_0 - I = 2.5 \times P(0,2.5) + 102.5 \times P(0,3) = 87.4730$ so that the put price is

$$0.8152 + 87.1222 - 87.4730 = 0.4644$$

Problem 30.11.

In the Hull–White model, $a = 0.08$ and $\sigma = 0.01$. Calculate the price of a one-year European call option on a zero-coupon bond that will mature in five years when the term structure is flat at 10%, the principal of the bond is \$100, and the strike price is \$68.

Using the notation in the text $P(0,T) = e^{-0.1\times 1} = 0.9048$ and $P(0,s) = e^{-0.1\times 5} = 0.6065$. Also

$$\sigma_P = \frac{0.01}{0.08}(1 - e^{-4\times 0.08})\sqrt{\frac{1 - e^{-2\times 0.08\times 1}}{2\times 0.08}} = 0.0329$$

and $h = -0.4192$ so that the call price is

$$100 \times 0.6065N(h) - 68 \times 0.9048N(h - \sigma_P) = 0.439$$

Problem 30.12.

Suppose that $a = 0.05$ and $\sigma = 0.015$ in the Hull–White model with the initial term structure being flat at 6% with semiannual compounding. Calculate the price of a 2.1-year European call option on a bond that will mature in three years. Suppose that the bond pays a coupon of 5% per annum semiannually. The principal of the bond is 100 and the strike price of the option is 99. The strike price is the cash price (not the quoted price) that will be paid for the bond.

The relevant parameters for the Hull–White model are $a = 0.05$ and $\sigma = 0.015$. Setting $\Delta t = 0.4$

$$\hat{B}(2.1,3) = \frac{B(2.1,3)}{B(2.1,2.5)} \times 0.4 = 0.88888$$

Also from equation (30.26), $\hat{A}(2.1,3) = 0.99925$ The first stage is to calculate the value of R at time 2.1 years which is such that the value of the bond at that time is 99. Denoting this value of R by R^*, we must solve

$$2.5e^{-R^*\times 0.4} + 102.5\hat{A}(2.1,3)e^{-\hat{B}(2.1,3)R^*} = 99$$

The solution to this for R^* turns out to be 6.626%. The option on the coupon bond is decomposed into an option with a strike price of 96.565 on a zero-coupon bond with a principal of 102.5 and an option with a strike price of 2.435 on a zero-coupon bond with a principal of 2.5. The first option is worth 0.0103 and the second option is worth 0.9343. The total value of the option is therefore 0.9446. (Note that the initial short rate with continuous compounding is 5.91%.)

Problem 30.13.

Use a change of numeraire argument to show that the relationship between the futures rate and forward rate for the Ho–Lee model is as shown in Section 6.4. Use the relationship to verify the expression for $\theta(t)$ given for the Ho–Lee model in equation (30.11)(Hint The futures price is a martingale when the market price of risk is zero. The forward price is a martingale when the market price of risk is a zero-coupon bond maturing at the same time as the forward contract.)

We will consider instantaneous forward and futures rates. (A more general result involving the forward and futures rate applying to a period of time between T_1 and T_2 is proved in Technical Note 1 on the author's site.)

Because $P(t,T) = A(t,T)e^{-r(T-t)}$ the process for $P(t,T)$ is from Itô's lemma

$$dP(t,T) = \ldots - \sigma(T-t)P(t,T)dz$$

Define $F(t,T)$ as the instantaneous forward rate for maturity T. The process for $F(0,T)$ is from Itô's lemma

$$dF(0,T) = \ldots + \sigma\, dz$$

The instantaneous forward rate with maturity T has a drift of zero in a world that is forward risk neutral with respect to $P(t,T)$. This is a world where the market price of risk is $-\sigma(T-t)$. When we move to a world where the market price of risk is zero the drift of the forward rate increases to $\sigma^2(T-t)$. Integrating this between $t=0$ and $t=T$ wee see that the forward rate grows by a total of $\sigma^2T^2/2$ between time 0 and time T in a world where the market price of risk is zero. The futures rate has zero growth rate in this world. At time T the forward rate equals the futures rate. It follows that the futures rate must exceed the forward rate by $\sigma^2T^2/2$ at time zero. This is consistent with the formula in Section 6.4.

Define $G(0,t)$ as the instantaneous futures rate for maturity t so that

$$G(0,t) - F(0,t) = \sigma^2t^2/2$$

and

$$G_t(0,t) - F_t(0,t) = \sigma^2t$$

In the traditional risk-neutral world the expected value of r at time t is the futures rate, $G(0,t)$. This means that the expected growth in r at time t must be $G_t(0,t)$ so that $\theta(t) = G_t(0,t)$. It follows that

$$\theta(t) = F_t(0,t) + \sigma^2t$$

This is equation (30.11).

Problem 30.14.

Use a similar approach to that in Problem 30.13 to derive the relationship between the futures rate and the forward rate for the Hull–White model. Use the relationship to verify the expression for $\theta(t)$ given for the Hull–White model in equation (30.14)

In this case we have $P(t,T) = A(t,T)e^{-B(t,T)r}$ so that from Itô's lemma

$$dP(t,T) = \ldots - \sigma B(t,T)P(t,T)dz$$

Define $F(t,T)$ as the instantaneous forward rate for maturity T. The process for $F(0,T)$ is from Itô's lemma

$$dF(0,T) = \ldots + \sigma e^{-a(T-t)}\,dz$$

This has drift of zero in a world that is forward risk neutral with respect to $P(t,T)$. This is a world where the market price of risk is $-\sigma B(t,T)$. When we move to a world where the market price of risk is zero the drift of $F(0,T)$ increases to $\sigma^2 e^{-a(T-t)}B(t,T)$. Integrating this between $t=0$ and $t=T$ wee see that the forward rate grows by a total of

$$\frac{\sigma^2}{2a^2}(1-e^{-aT})^2$$

between time 0 and time T in a world where the market price of risk is zero. The futures price has zero growth rate in this world. At time T the forward price equals the futures price. It follows that the futures price must exceed the forward price by

$$\frac{\sigma^2}{2a^2}(1-e^{-aT})^2$$

at time zero.[3]

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[3] To produce a result relating the futures rate for the period between times T_1 and T_2 to the forward rate between this period we can proceed as in Technical Note 1 on the author's web site. The drift of the forward rate is

$$\frac{\sigma^2 B(t,T_2)^2 - \sigma^2 B(t,T_1)^2}{2(T_2-T_1)}$$

$$= \frac{\sigma^2}{2a^2(T_2-T_1)}[e^{at}(-2e^{-aT_2}+2e^{-aT_1}) + e^{2at}[e^{-2aT_2} - e^{-2aT_1}]$$

Integrating between time 0 and time T_1 we get

$$\frac{\sigma^2}{2a^2(T_2-T_1)}[(e^{aT_1}-1)(-2e^{-aT_2}+2e^{-aT_1})/a + (e^{2aT_1}-1)(e^{-2aT_2}-e^{-2aT_1})/(2a)]$$

$$= \frac{\sigma^2 B(T_1,T_2)}{4a^2(T_2-T_1)}[4(1-e^{-aT_1}) - (1-e^{-2aT_1})(1+e^{-a(T_2-T_1)})]$$

$$= \frac{B(T_1,T_2)}{T_2-T_1}[B(T_1,T_2)(1-e^{-2aT_1}) + 2aB(0,T_1)^2]\frac{\sigma^2}{4a}$$

This is the amount by which the futures rate exceeds the forward rate at time zero.

Define $G(0,t)$ as the instantaneous futures rate for maturity t so that

$$G(0,t) - F(0,t) = \frac{\sigma^2}{2a^2}(1 - e^{-at})^2$$

and

$$G_t(0,t) - F_t(0,t) = \frac{\sigma^2}{a}(1 - e^{-at})e^{-at}$$

In the traditional risk-neutral world the expected value of r at time t is the futures rate, $G(0,t)$. This means that the expected growth in r at time t must be $G_t(0,t) - a[r - G(0,t)]$ so that $\theta(t) - ar = G_t(0,t) - a[r - G(0,t)]$. It follows that

$$\theta(t) = G_t(0,t) + aG(0,t)$$

$$= F_t(0,t) + aF(0,t) + \frac{\sigma^2}{a}(1 - e^{-at})e^{-at} + \frac{\sigma^2}{2a}(1 - e^{-at})^2$$

$$= F_t(0,t) + aF(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

This proves equation (30.14).

Problem 30.15.

Suppose that $a = 0.05$, $\sigma = 0.015$, and the term structure is flat at 10%. Construct a trinomial tree for the Hull–White model where there are two time steps, each one year in length.

The time step, Δt, is 1 so that $\Delta r = 0.015\sqrt{3} = 0.02598$. Also $j_{\max} = 4$ showing that the branching method should change four steps from the center of the tree. With only three steps we never reach the point where the branching changes. The tree is shown in Figure S30.1.

Problem 30.16.

Calculate the price of a two-year zero-coupon bond from the tree in Figure 30.6.

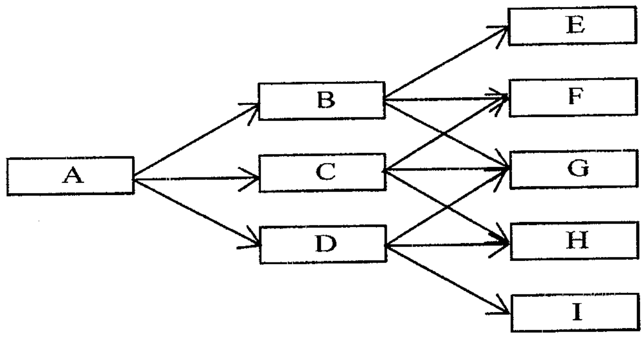
A two-year zero-coupon bond pays off \$100 at the ends of the final branches. At node B it is worth $100e^{-0.12\times1} = 88.69$. At node C it is worth $100e^{-0.10\times1} = 90.48$. At node D it is worth $100e^{-0.08\times1} = 92.31$. It follows that at node A the bond is worth

$$(88.69 \times 0.25 + 90.48 \times 0.5 + 92.31 \times 0.25)e^{-0.1\times1} = 81.88$$

or \$81.88

Problem 30.17.

Calculate the price of a two-year zero-coupon bond from the tree in Figure 30.9 and verify that it agrees with the initial term structure.



Node	A	B	C	D	E	F	G	H	I
r	10.00%	12.61%	10.01%	7.41%	15.24%	12.64%	10.04%	7.44%	4.84%
p_u	0.1667	0.1429	0.1667	0.1929	0.1217	0.1429	0.1667	0.1929	0.2217
p_m	0.6666	0.6642	0.6666	0.6642	0.6567	0.6642	0.6666	0.6642	0.6567
p_d	0.1667	0.1929	0.1667	0.1429	0.2217	0.1929	0.1667	0.1429	0.1217

Figure S30.1 Tree for Problem 30.15.

A two-year zero-coupon bond pays off \$100 at time two years. At node B it is worth $100e^{-0.0693\times 1} = 93.30$. At node C it is worth $100e^{-0.0520\times 1} = 94.93$. At node D it is worth $100e^{-0.0347\times 1} = 96.59$. It follows that at node A the bond is worth

$$(93.30 \times 0.167 + 94.93 \times 0.666 + 96.59 \times 0.167)e^{-0.0382\times 1} = 91.37$$

or \$91.37. Because $91.37 = 100e^{-0.04512\times 2}$, the price of the two-year bond agrees with the initial term structure.

Problem 30.18.

Calculate the price of an 18-month zero-coupon bond from the tree in Figure 30.10 and verify that it agrees with the initial term structure.

An 18-month zero-coupon bond pays off \$100 at the final nodes of the tree. At node E it is worth $100e^{-0.088\times 0.5} = 95.70$. At node F it is worth $100e^{-0.0648\times 0.5} = 96.81$. At node G it is worth $100e^{-0.0477\times 0.5} = 97.64$. At node H it is worth $100e^{-0.0351\times 0.5} = 98.26$. At node I it is worth $100e^{0.0259\times 0.5} = 98.71$. At node B it is worth

$$(0.118 \times 95.70 + 0.654 \times 96.81 + 0.228 \times 97.64)e^{-0.0564\times 0.5} = 94.17$$

Similarly at nodes C and D it is worth 95.60 and 96.68. The value at node A is therefore

$$(0.167 \times 94.17 + 0.666 \times 95.60 + 0.167 \times 96.68)e^{-0.0343 \times 0.5} = 93.92$$

The 18-month zero rate is $0.08 - 0.05e^{-0.18 \times 1.5} = 0.0418$. This gives the price of the 18-month zero-coupon bond as $100e^{-0.0418 \times 1.5} = 93.92$ showing that the tree agrees with the initial term structure.

Problem 30.19.

What does the calibration of a one-factor term structure model involve?

The calibration of a one-factor interest rate model involves determining its volatility parameters so that it matches the market prices of actively traded interest rate options as closely as possible.

Problem 30.20.

Use the DerivaGem software to value 1×4, 2×3, 3×2, and 4×1 European swap options to receive fixed and pay floating. Assume that the one, two, three, four, and five year interest rates are 6%, 5.5%, 6%, 6.5%, and 7%, respectively. The payment frequency on the swap is semiannual and the fixed rate is 6% per annum with semiannual compounding. Use the Hull–White model with $a = 3\%$ and $\sigma = 1\%$. Calculate the volatility implied by Black's model for each option.

The option prices are 0.1302, 0.0814, 0.0580, and 0.0274. The implied Black volatilities are 14.28%, 13.64%, 13.24%, and 12.81%

Problem 30.21.

Prove equations (30.25), (30.26), and (30.27).

From equation (30.15)

$$P(t, t + \Delta t) = A(t, t + \Delta t)e^{-r(t)B(t,t+\Delta t)}$$

Also

$$P(t, t + \Delta t) = e^{-R(t)\Delta t}$$

so that

$$e^{-R(t)\Delta t} = A(t, t + \Delta t)e^{-r(t)B(t,t+\Delta t)}$$

or

$$e^{-r(t)B(t,T)} = \frac{e^{-R(t)B(t,T)\Delta t/B(t,t+\Delta t)}}{A(t, t + \Delta t)^{B(t,T)/B(t,t+\Delta t)}}$$

Hence equation (30.25) is true with

$$\hat{B}(t, T) = \frac{B(t, T)\Delta t}{B(t, t + \Delta t}$$

and

$$\hat{A}(t,T) = \frac{A(t,T)}{A(t,t+\Delta t)^{B(t,T)/B(t,t+\Delta t)}}$$

or

$$\ln \hat{A}(t,T) = \ln A(t,T) - \frac{B(t,T)}{B(t,t+\Delta t)} \ln A(t,t+\Delta t)$$

ASSIGNMENT QUESTIONS

Problem 30.22.

Construct a trinomial tree for the Ho and Lee model where $\sigma = 0.02$. Suppose that the the initial zero-coupon interest rate for a maturities of 0.5, 1.0, and 1.5 years are 7.5%, 8%, and 8.5%. Use two time steps, each six months long. Calculate the value of a zero-coupon bond with a face value of \$100 and a remaining life of six months at the ends of the final nodes of the tree. Use the tree to value a one-year European put option with a strike price of 95 on the bond. Compare the price given by your tree with the analytic price given by DerivaGem.

The tree is shown in Figure M30.1. The probability on each upper branch is 1/6; the probability on each middle branch is 2/3; the probability on each lower branch is 1/6. The six month bond prices nodes E, F, G, H, I are $100e^{-0.1442\times0.5}$, $100e^{-0.1197\times0.5}$, $100e^{-0.0952\times0.5}$, $100e^{-0.0707\times0.5}$, and $100e^{-0.0462\times0.5}$, respectively. These are 93.04, 94.19, 95.35, 96.53, and 97.71. The payoffs from the option at nodes E, F, G, H, and I are therefore 1.96, 0.81, 0, 0, and 0. The value at node B is $(0.1667\times1.96+0.6667\times0.81)e^{-0.1095\times0.5} = 0.8380$. The value at node C is $0.1667\times0.81\times e^{-0.0851\times0.5} = 0.1294$. The value at node D is zero. The value at node A is

$$(0.1667\times0.8380+0.6667\times0.1294)e^{-0.0750\times0.5} = 0.217$$

The answer given by DerivaGem using the analytic approach is 0.213.

Problem 30.23.

A trader wishes to compute the price of a one-year American call option on a five-year bond with a face value of 100. The bond pays a coupon of 6% semiannually and the (quoted) strike price of the option is \$100. The continuously compounded zero rates for maturities of six months, one year, two years, three years, four years, and five years are 4.5%, 5%, 5.5%, 5.8%, 6.1%, and 6.3%. The best fit reversion rate for either the normal or the lognormal model has been estimated as 5%.

A one year European call option with a (quoted) strike price of 100 on the bond is actively traded. Its market price is \$0.50. The trader decides to use this option for calibration. Use the DerivaGem software with ten time steps to answer the following questions.

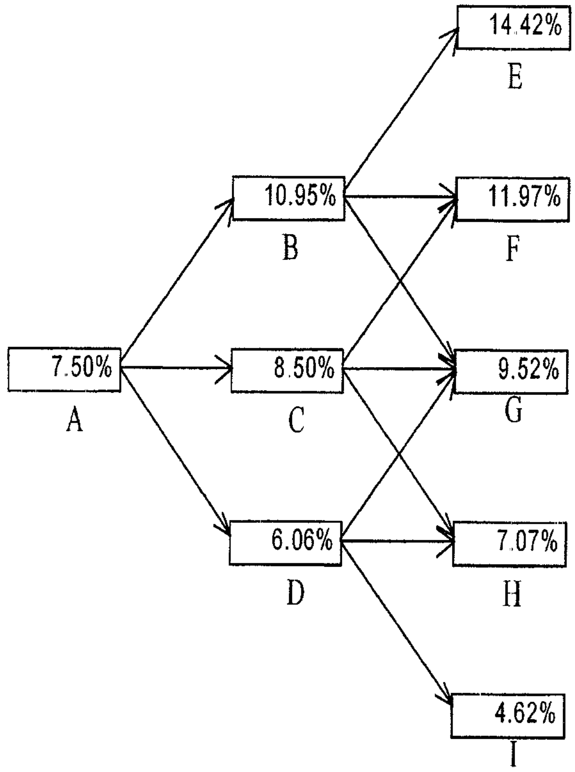


Figure M30.1 Tree for Problem 30.22

(a) Assuming a normal model, imply the σ parameter from the price of the European option.
(b) Use the σ parameter to calculate the price of the option when it is American.
(c) Repeat (a) and (b) for the lognormal model. Show that the model used does not significantly affect the price obtained providing it is calibrated to the known European price.
(d) Display the tree for the normal model and calculate the probability of a negative interest rate occurring.
(e) Display the tree for the lognormal model and verify that the option price is correctly calculated at the node where, with the notation of Section 30.7, $i = 9$ and $j = -1$.

(a) The implied value of σ is 1.12%.

(b) The value of the American option is 0.593

(c) The implied value of σ is 18.49% and the value of the American option is 0.593. The two models give the same answer providing they are both calibrated to the same European price.

(d) We get a negative interest rate if there are 10 down moves. The probability of this is $0.16667 \times 0.16418 \times 0.16172 \times 0.15928 \times 0.15687 \times 0.15448 \times 0.15218 \times 0.14978 \times 0.14747 \times 0.14518 = 8.3 \times 10^{-9}$

(e) The calculation is

$$0.164179 \times 1.705358 \times e^{-0.052571 \times 0.1} = 0.278516$$

Problem 30.24.

Use the DerivaGem software to value 1×4, 2×3, 3×2, *and* 4×1 *European swap options to receive floating and pay fixed. Assume that the one, two, three, four, and five year interest rates are 3%, 3.5%, 3.8%, 4.0%, and 4.1%, respectively. The payment frequency on the swap is semiannual and the fixed rate is 4% per annum with semiannual compounding. Use the lognormal model with* $a = 5\%$, $\sigma = 15\%$, *and 50 time steps. Calculate the volatility implied by Black's model for each option.*

The values of the four European swap options are 1.72, 1.73, 1.30, and 0.65, respectively. The implied Black volatilities are 13.37%, 13.34%, 13.43%, and 13.42%, respectively.

Problem 30.25.

Verify that the DerivaGem software gives Figure 30.11 for the example considered. Use the software to calculate the price of the American bond option for the lognormal and normal models when the strike price is 95, 100, and 105. In the case of the normal model, assume that $a = 5\%$ *and* $\sigma = 1\%$. *Discuss the results in the context of the heaviness of the tails arguments of Chapter 18.*

With 100 time steps the lognormal model gives prices of 5.569, 2.433, and 0.699 for strike prices of 95, 100, and 105. With 100 time steps the normal model gives prices of 5.493, 2.511, and 0.890 for the three strike prices respectively. The normal model gives a heavier left tail and a less heavy right tail than the lognormal model for interest rates. This translates into a less heavy left tail and a heavier right tail for bond prices. The arguments in Chapter 18 show that we expect the normal model to give higher option prices for high strike prices and lower option prices for low strike. This is indeed what we find.

Problem 30.26.

Modify Sample Application G in the DerivaGem Application Builder software to test the convergence of the price of the trinomial tree when it is used to price a two-year call option on a five-year bond with a face value of 100. Suppose that the strike price (quoted) is 100, the coupon rate is 7% with coupons being paid twice a year. Assume that the zero curve is as in Table 30.2. Compare results for the following cases:

a. *Option is European; normal model with* $\sigma = 0.01$ *and* $a = 0.05$.
b. *Option is European; lognormal model with* $\sigma = 0.15$ *and* $a = 0.05$.
c. *Option is American; normal model with* $\sigma = 0.01$ *and* $a = 0.05$.
d. *Option is American; lognormal model with* $\sigma = 0.15$ *and* $a = 0.05$.

The results are shown in Figure M30.2.

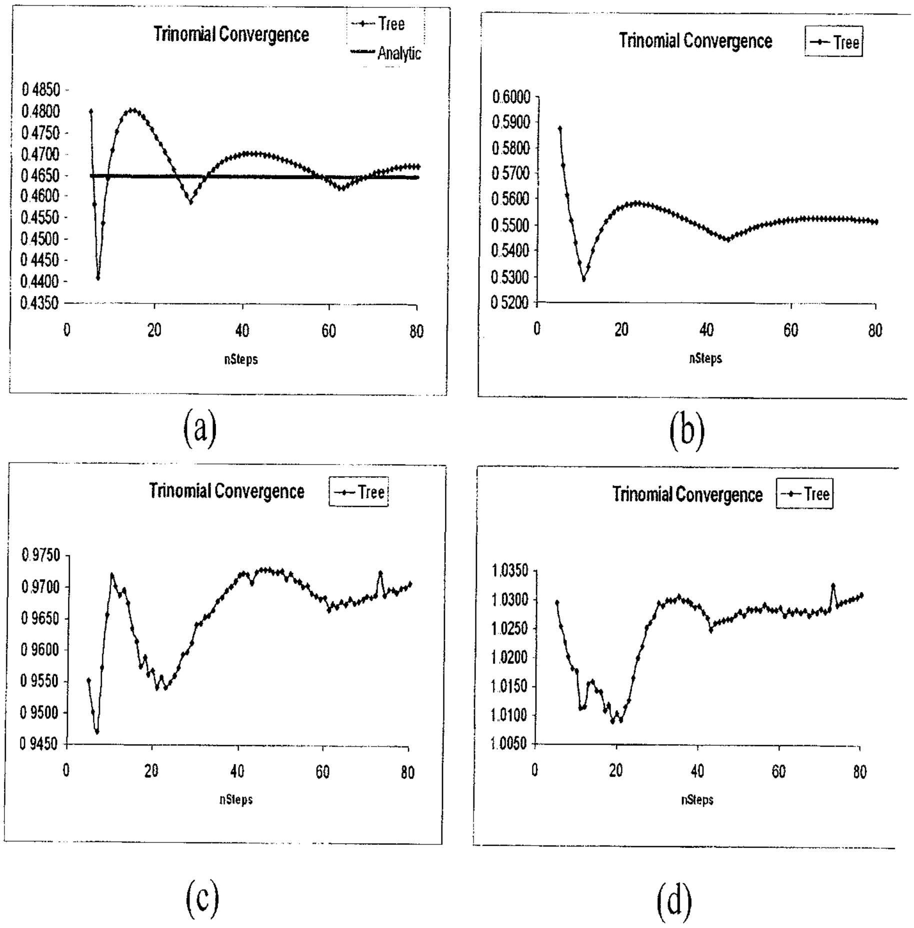


Figure M30.2 Tree for Problem 30.26

CHAPTER 31

Interest Rate Derivatives: HJM and LMM

Notes for the Instructor

This chapter deals with the Heath, Jarrow, and Morton model and Libor Market Model model. (The latter is sometimes also referred to as the Brace, Gatarek, and Musiela or BGM model). This chapter is more mathematically challenging than most other chapters in the book. Instructors who are not teaching advanced students will probably choose to skip it. The chapter deals with multifactor no-arbitrage term structure models. A final section gives some information about the mortgage-backed security market in the United States and option-adjusted spreads. (This section can be taught without the rest of the chapter being covered.)

In addition doing the assignments in the text, the most advanced students can be asked to implement the HJM or LIBOR market model and use it to value some of the mortgage-backed securities described in Section 31.3.

QUESTIONS AND PROBLEMS

Problem 31.1.

Explain the difference between a Markov and a non-Markov model of the short rate.

In a Markov model the expected change and volatility of the short rate at time t depend only on the value of the short rate at time t. In a non-Markov model they depend on the history of the short rate prior to time t.

Problem 31.2.

Prove the relationship between the drift and volatility of the forward rate for the multifactor version of HJM in equation (31.6).

Equation (31.1) becomes

$$dP(t,T) = r(t)P(t,T)\,dt + \sum_k v_k(t,T,\Omega_t)P(t,T)\,dz_k(t)$$

so that

$$d\ln[P(t,T_1)] = \left[r(t) - \sum_k \frac{v_k(t,T_1,\Omega_t)^2}{2}\right] dt + \sum_k v_k(t,T_1,\Omega_t)\,dz_k(t)$$

and

$$d\ln[P(t,T_2)] = \left[r(t) - \sum_k \frac{v_k(t,T_2,\Omega_t)^2}{2}\right] dt + v_k(t,T_2,\Omega_t)\, dz_k(t)$$

From equation (31.2)

$$df(t,T_1,T_2) = \frac{\sum_k [v_k(t,T_2,\Omega_t)^2 - v_k(t,T_1,\Omega_t)^2]}{2(T_2-T_1)} dt + \sum_k \frac{v_k(t,T_1,\Omega_t) - v_k(t,T_2,\Omega_t)}{T_2 - T_1} dz_k(t)$$

Putting $T_1 = T$ and $T_2 = T + \Delta t$ and taking limits as Δt tends to zero this becomes

$$dF(t,T) = \sum_k \left[v_k(t,T,\Omega_t)\frac{\partial v_k(t,T,\Omega_t)}{\partial T}\right] dt - \sum_k \left[\frac{\partial v_k(t,T,\Omega_t)}{\partial T}\right] dz_k(t)$$

Using $v_k(t,t,\Omega_t) = 0$

$$v_k(t,T,\Omega_t) = \int_t^T \frac{\partial v_k(t,\tau,\Omega_t)}{\partial \tau} d\tau$$

The result in equation (31.6) follows by substituting

$$s_k(t,T,\Omega_t) = \frac{\partial v_k(t,T,\Omega_t)}{\partial T}$$

Problem 31.3.

"When the forward rate volatility $s(t,T)$ in HJM is constant, the Ho–Lee model results." Verify that this is true by showing that HJM gives a process for bond prices that is consistent with the Ho–Lee model in Chapter 30.

Using the notation in Section 31.1, when s is constant,

$$v_T(t,T) = s \qquad v_{TT}(t,T) = 0$$

Integrating $v_T(t,T)$

$$v(t,T) = sT + \alpha(t)$$

for some function α. Using the fact that $v(T,T) = 0$, we must have

$$v(t,T) = s(T-t)$$

Using the notation from Chapter 30, in Ho–Lee $P(t,T) = A(t,T)e^{-r(T-t)}$. The standard deviation of the short rate is constant. It follows from Itô's lemma that the standard deviation of the bond price is a constant times the bond price times $T-t$. The volatility of the bond price is therefore a constant times $T-t$. This shows that Ho–Lee is consistent with a constant s.

Problem 31.4.

"When the forward rate volatility, $s(t,T)$ in HJM is $\sigma e^{-a(T-t)}$ the Hull–White model results." Verify that this is true by showing that HJM gives a process for bond prices that is consistent with the Hull–White model in Chapter 30.

Using the notation in Section 31.1, when $v_T(t,T) = s(t,T) = \sigma e^{-a(T-t)}$

$$v_{TT}(t,T) = -a\sigma e^{-a(T-t)}$$

Integrating $v_T(t,T)$

$$v(t,T) = -\frac{1}{a}\sigma e^{-a(T-t)} + \alpha(t)$$

for some function α. Using the fact that $v(T,T) = 0$, we must have

$$v(t,T) = \frac{\sigma}{a}[1 - e^{-a(T-t)}] = \sigma B(t,T)$$

Using the notation from Chapter 30, in Hull–White $P(t,T) = A(t,T)e^{-rB(t,T)}$. The standard deviation of the short rate is constant, σ. It follows from Itô's lemma that the standard deviation of the bond price is $\sigma P(t,T)B(t,T)$. The volatility of the bond price is therefore $\sigma B(t,T)$. This shows that Hull-White is consistent with $s(t,T) = \sigma e^{-a(T-t)}$.

Problem 31.5.

What is the advantage of LMM over HJM?

LMM is a similar model to HJM. It has the advantage over HJM that it involves forward rates that are readily observable. HJM involves instantaneous forward rates.

Problem 31.6.

Provide an intuitive explanation of why a ratchet cap increases in value as the number of factors increase.

A ratchet cap tends to provide relatively low payoffs if a high (low) interest rate at one reset date is followed by a high (low) interest rate at the next reset date. High payoffs occur when a low interest rate is followed by a high interest rate. As the number of factors increase, the correlation between successive forward rates declines and there is a greater chance that a low interest rate will be followed by a high interest rate.

Problem 31.7.

Show that equation (31.10) reduces to (31.4) as the δ_i tend to zero.

Equation (31.10) can be written

$$dF_k(t) = \zeta_k(t)F_k(t) \sum_{i=m(t)}^{k} \frac{\delta_i F_i(t)\zeta_i(t)}{1 + \delta_i F_i(t)} \, dt + \zeta_k(t)F_k(t) \, dz$$

As δ_i tends to zero, $\zeta_i(t)F_i(t)$ becomes the standard deviation of the instantaneous t_i-maturity forward rate at time t. Using the notation of Section 31.1 this is $s(t, t_i, \Omega_t)$. As δ_i tends to zero

$$\sum_{i=m(t)}^{k} \frac{\delta_i F_i(t)\zeta_i(t)}{1+\delta_i F_i(t)}$$

tends to

$$\int_{\tau=t}^{t_k} s(t, \tau, \Omega_t)\, d\tau$$

Equation (31.10) therefore becomes

$$dF_k(t) = \left[s(t, t_k, \Omega_t) \int_{\tau=t}^{t_k} s(t, \tau, \Omega_t)\, d\tau \right] dt + s(t, t_k, \Omega_t)\, dz$$

This is the HJM result.

Problem 31.8.

Explain why a sticky cap is more expensive than a similar ratchet cap.

In a ratchet cap, the cap rate equals the previous reset rate, R, plus a spread. In the notation of the text it is $R_j + s$. In a sticky cap the cap rate equal the previous capped rate plus a spread. In the notation of the text it is $\min(R_j, K_j) + s$. The cap rate in a ratchet cap is always at least a great as that in a sticky cap. Since the value of a cap is a decreasing function of the cap rate, it follows that a sticky cap is more expensive.

Problem 31.9.

Explain why IOs and POs have opposite sensitivities to the rate of prepayments

When prepayments increase, the principal is received sooner. This increases the value of a PO. When prepayments increase, less interest is received. This decreases the value of an IO.

Problem 31.10.

An option adjusted spread is analogous to the yield on a bond." Explain this statement.

A bond yield is the discount rate that causes the bond's price to equal the market price. The same discount rate is used for all maturities. An OAS is the parallel shift to the Treasury zero curve that causes the price of an instrument such as a mortgage-backed security to equal its market price.

Problem 31.11.

Prove equation (31.15).

When there are p factors equation (31.7) becomes

$$dF_k(t) = \sum_{q=1}^{p} \zeta_{k,q}(t)F_k(t)dz_q$$

Equation (31.8) becomes

$$dF_k(t) = \sum_{q=1}^{p} \zeta_{k,q}(t)[v_{m(t),q} - v_{k+1,q}]F_k(t)dt + \sum_{q=1}^{p} \zeta_{k,q}(t)(F_k(t)dz_q$$

Equation coefficients of dz_q in

$$\ln P(t, t_i) - \ln P(t, t_{i+1}) = \ln[1 + \delta_i F_i(t)]$$

Equation (31.9) therefore becomes

$$v_{i,q}(t) - v_{i+1,q}(t) = \frac{\delta_i F_i(t)\zeta_{i,q}}{1 + \delta_i F_i(t)}$$

Equation (31.15) follows.

Problem 31.12.

Prove the formula for the variance, $V(T)$, of the swap rate in equation (31.17).

From the equations in the text

$$s(t) = \frac{P(t, T_0) - P(t, T_N)}{\sum_{i=0}^{N-1} \tau_i P(t, T_{i+1})}$$

and

$$\frac{P(t, T_i)}{P(t, T_0)} = \prod_{j=0}^{i-1} \frac{1}{1 + \tau_j G_j(t)}$$

so that

$$s(t) = \frac{1 - \prod_{j=0}^{N-1} \frac{1}{1+\tau_j G_j(t)}}{\sum_{i=0}^{N-1} \tau_i \prod_{j=0}^{i} \frac{1}{1+\tau_j G_j(t)}}$$

(We employ the convention that empty sums equal zero and empty products equal one.) Equivalently

$$s(t) = \frac{\prod_{j=0}^{N-1}[1 + \tau_j G_j(t)] - 1}{\sum_{i=0}^{N-1} \tau_i \prod_{j=i+1}^{N-1}[1 + \tau_j G_j(t)]}$$

or

$$\ln s(t) = \ln\left\{\prod_{j=0}^{N-1}[1 + \tau_j G_j(t)] - 1\right\} - \ln\left\{\sum_{i=0}^{N-1} \tau_i \prod_{j=i+1}^{N-1}[1 + \tau_j G_j(t)]\right\}$$

so that

$$\frac{1}{s(t)}\frac{\partial s(t)}{\partial G_k(t)} = \frac{\tau_k \gamma_k(t)}{1 + \tau_k G_k(t)}$$

where

$$\gamma_k(t) = \frac{\prod_{j=0}^{N-1}[1+\tau_j G_j(t)]}{\prod_{j=0}^{N-1}[1+\tau_j G_j(t)] - 1} - \frac{\sum_{i=0}^{k-1}\tau_i \prod_{j=i+1}^{N-1}[1+\tau_j G_j(t)]}{\sum_{i=0}^{N-1}\tau_i \prod_{j=i+1}^{N-1}[1+\tau_j G_j(t)]}$$

From Ito's lemma the qth component of the volatility of $s(t)$ is

$$\sum_{k=0}^{N-1} \frac{1}{s(t)} \frac{\partial s(t)}{\partial G_k(t)} \beta_{k,q}(t) G_k(t)$$

or

$$\sum_{k=0}^{N-1} \frac{\tau_k \beta_{k,q}(t) G_k(t) \gamma_k(t)}{1+\tau_k G_k(t)}$$

The variance rate of $s(t)$ is therefore

$$V(t) = \sum_{q=1}^{p} \left[\sum_{k=0}^{N-1} \frac{\tau_k \beta_{k,q}(t) G_k(t) \gamma_k(t)}{1+\tau_k G_k(t)} \right]^2$$

Problem 31.13.

Prove equation (31.19).

$$1+\tau_j G_j(t) = \prod_{m=1}^{M} [1+\tau_{j,m} G_{j,m}(t)]$$

so that

$$\ln[1+\tau_j G_j(t)] = \sum_{m=1}^{M} \ln[1+\tau_{j,m} G_{j,m}(t)]$$

Equating coefficients of dz_q

$$\frac{\tau_j \beta_{j,q}(t) G_j(t)}{1+\tau_j G_j(t)} = \sum_{m=1}^{M} \frac{\tau_{j,m} \beta_{j,m,q}(t) G_{j,m}(t)}{1+\tau_{j,m} G_{j,m}(t)}$$

If we assume that $G_{j,m}(t) = G_{j,m}(0)$ for the purposes of calculating the swap volatility we see from equation (31.17) that the volatility becomes

$$\sqrt{\frac{1}{T_0} \int_{t=0}^{T_0} \sum_{q=1}^{p} \left[\sum_{k=n}^{N-1} \sum_{m=1}^{M} \frac{\tau_{k,m} \beta_{k,m,q}(t) G_{k,m}(0) \gamma_k(0)}{1+\tau_{k,m} G_{k,m}(0)} \right]^2 dt}$$

This is equation (31.19).

Problem 31.14.

In an annual-pay cap the Black volatilities for caplets with maturities one, two, three, and five years are 18%, 20%, 22%, and 20%, respectively. Estimate the volatility of a one-year forward rate in the LIBOR Market Model when the time to maturity is (a) zero to one year, (b) one to two years, (c) two to three years, and (d) three to five years. Assume that the zero curve is flat at 5% per annum (annually compounded). Use DerivaGem to estimate flat volatilities for two-, three-, four-, five-, and six-year caps.

The cumulative variances for one, two, three, and five years are $0.18^2 \times 1 = 0.0324$, $0.2^2 \times 2 = 0.08$, $0.22^2 \times 3 = 0.1452$, and $0.2^2 \times 5 = 0.2$, respectively. If the required forward rate volatilities are Λ_1, Λ_2, Λ_3, and Λ_4, we must have

$$\Lambda_1^2 = 0.0324$$

$$\Lambda_2^2 \times 1 = 0.08 - 0.0324$$

$$\Lambda_3^2 \times 1 = 0.1452 - 0.08$$

$$\Lambda_4^2 \times 2 = 0.2 - 0.1452$$

It follows that $\Lambda_1 = 0.18$, $\Lambda_2 = 0.218$, $\Lambda_3 = 0.255$, and $\Lambda_4 = 0.166$

For the last part of the question we first interpolate to obtain the spot volatility for the four-year caplet as 21%. The yield curve is flat at 4.879% with continuous compounding. We use DerivaGem to calculate the prices of caplets with a strike price of 5% where the underlying option matures in one, two, three, four, and five years. The results are 0.3252, 0.4857, 0.6216, 0.6516, and 0.6602, respectively. This means that the prices two-, three-, four-, five-, and six-year caps are 0.3252, 0.8109, 1.4325, 2.0841, and 2.7443. We use DerivaGem again to imply flat volatilities from these prices. The flat volatilities for two-, three-, four-, five-, and six-year caps are 18%, 19.14%, 20.28%, 20.49%, and 20.37%, respectively

Problem 31.15.

In the flexi cap considered in Section 31.2 the holder is obligated to exercise the first N in-the-money caplets. After that no further caplets can be exercised. (In the example, $N = 5$.) Two other ways that flexi caps are sometimes defined are:

a. *The holder can choose whether any caplet is exercised, but there is a limit of N on the total number of caplets that can be exercised.*
b. *Once the holder chooses to exercise a caplet all subsequent in-the-money caplets must be exercised up to a maximum of N.*

Discuss the problems in valuing these types of flexi caps. Of the three types of flexi caps, which would you expect to be most expensive? Which would you expect to be least expensive?

The two types of flexi caps mentioned are more difficult to value than the flexi cap considered in Section 31.2. There are two reasons for this.

(i) They are American-style. (The holder gets to choose whether a caplet is exercised.) This makes the use of Monte Carlo simulation difficult.

(ii) They are path dependent. In (a) the decision on whether to exercise a caplet is liable to depend on the number of caplets exercised so far. In (b) the exercise of a caplet is liable to depend on a decision taken some time earlier.

In practice, flexi caps are sometimes valued using a one-factor model of the short rate in conjunction with the techniques described in Section 26.5 for handling path-dependent derivatives.

The flexi cap in (b) is worth more than the flexi cap considered in Section 31.2. This is because the holder of the flexi cap in (b) has all the options of the holder of the flexi cap in the text and more. Similarly the flexi cap in (a) is worth more than the flexi cap in (b). This is because the holder of the flexi cap in (a) has all the options of the holder of the flexi cap in (b) and more. We therefore expect the flexi cap in (a) to be the most expensive and the flexi cap considered in section 31.2 to be the least expensive.

CHAPTER 32

Swaps Revisited

Notes for the Instructor

This chapter describes a number of nonstandard swap products. These include compounding swaps, currency swaps, LIBOR-in-arrears swaps, CMS and CMT swaps, differential (diff) swaps, equity swaps, accrual swaps, cancelable swaps, index amortizing swaps, commodity swaps, and volatility swaps. One of the aims of the chapter is to distinguish between swaps where the "assume forward rates are realized" rule can be used from swaps where it cannot be used. LIBOR-in-arrears swaps, CMS and CMT swaps, and diff swaps are examples of swaps where the rule cannot be used. As shown in the chapter, the valuation of these deals depend on the convexity, timing, and quanto adjustments explained in Chapters 29.

If Chapter 29 has not been covered it is still possible to teach this chapter by skipping the technical valuation issues. Problem 32.10 and 32.12 can then be used as assignment questions. If the more technical material has been covered Problems 32.9 and 32.11 can be assigned.

QUESTIONS AND PROBLEMS

Problem 32.1.

Calculate all the fixed cash flows and their exact timing for the swap in Business Snapshot 32.1. Assume that the day count conventions are applied using target payment dates rather than actual payment dates.

The target payment dates are July 11, 2007; January 11, 2008; July 11, 2008; January 11, 2009; July 11, 2009; January 11, 2010; July 11, 2010; January 11, 2011; July 11, 2011; January 11, 2012. These occur on Wednesday, Friday, Friday, Sunday, Saturday, Monday, Sunday, Tuesday, Monday, and Wednesday, respectively. The actual payment dates are therefore July 11, 2007; January 11, 2008; July 11, 2008; January 12, 2009, July 13, 2009, January 11, 2010, July 12, 2010, January 11, 2011, July 11, 2011, and January 11, 2012. The fixed rate day count convention is Actual/365. There are 181 days between January 11, 2007 and July 11, 2007. This means that the fixed payments on July 11, 2007 is

$$\frac{181}{365} \times 0.06 \times 100,000,000 = \$2,975,342$$

Similarly subsequent fixed cash flows are: \$3,024,658, \$2,991,781 \$3,041,096, \$2,991,781, \$2,991,781, \$2,991,781, \$3,008,219, \$2,975,342, and \$3,024,658.

Problem 32.2.

Suppose that a swap specifies that a fixed rate is exchanged for twice the LIBOR rate. Can the swap be valued using the "assume forward rates are realized" rule?

Yes. The swap is the same as one on twice the principal where half the fixed rate is exchanged for the LIBOR rate.

Problem 32.3.

What is the value of a two-year fixed-for-floating compound swap where the principal is \$100 million and payments are made semiannually? Fixed interest is received and floating is paid. The fixed rate is 8% and it is compounded at 8.3% (both semiannually compounded). The floating rate is LIBOR plus 10 basis points and it is compounded at LIBOR plus 20 basis points. The LIBOR zero curve is flat at 8% with semiannual compounding.

The final fixed payment is in millions of dollars:

$$[(4 \times 1.0415 + 4) \times 1.0415 + 4] \times 1.0415 + 4 = 17.0238$$

The final floating payment assuming forward rates are realized is

$$[(4.05 \times 1.041 + 4.05) \times 1.041 + 4.05] \times 1.041 + 4.05 = 17.2238$$

The value of the swap is therefore $-0.2000/(1.04^4) = -0.1710$ or –\$171,000.

Problem 32.4.

What is the value of a five-year swap where LIBOR is paid in the usual way and in return LIBOR compounded at LIBOR is received on the other side? The principal on both sides is \$100 million. Payment dates on the pay side and compounding dates on the receive side are every six months and the yield curve is flat at 5% with semiannual compounding.

The value is zero. The receive side is the same as the pay side with the cash flows compounded forward at LIBOR. Compounding cash flows forward at LIBOR does not change their value.

Problem 32.5.

Explain carefully why a bank might choose to discount cash flows on a currency swap at a rate slightly different from LIBOR.

In theory, a new floating-for-floating swap should involve exchanging LIBOR in one currency for LIBOR in another currency (with no spreads added). In practice, macroeconomic effects give rise to spreads. Financial institutions often adjust the discount rates they use to allow for this. Suppose that USD LIBOR is always exchanged Swiss franc LIBOR plus 15 basis points. Financial institutions would discount USD cash flows at USD LIBOR and Swiss franc cash flows at LIBOR plus 15 basis points. This would ensure that the floating-for-floating swap is valued consistently with the market.

Problem 32.6.

Calculate the total convexity/timing adjustment in Example 32.3 of Section 32.4 if all cap volatilities are 18% instead of 20% and volatilities for all options on five-year swaps are 13% instead of 15%. What should the five year swap rate in three years time be assumed for the purpose of valuing the swap? What is the value of the swap?

In this case $y_i = 0.05$, $\sigma_{y,i} = 0.13$, $\tau_i = 0.5$, $F_i = 0.05$, $\sigma_{F,i} = 0.18$, and $\rho_i = 0.7$ for all i. It is still true that $G_i'(y_i) = -437.603$ and $G_i''(y_i) = 2261.23$. Equation (32.2) gives the total convexity/timing adjustment as $0.0000892t_i$ or 0.892 basis points per year until the swap rate is observed. The swap rate in three years should be assumed to be 5.0268%. The value of the swap is \$119,069.

Problem 32.7.

Explain why a plain vanilla interest rate swap and the compounding swap in Section 32.2 can be valued using the "assume forward rates are realized" rule, but a LIBOR-in-arrears swap in Section 32.4 cannot.

In a plain vanilla swap we can enter into a series of FRAs to exchange the floating cash flows for their values if the "assume forward rates are realized rule" is used. In the case of a compounding swap Section 32.2 shows that we are able to enter into a series of FRAs that exchange the final floating rate cash flow for its value when the "assume forward rates are realized rule" is used. There is no way of entering into FRAs so that the floating-rate cash flows in a LIBOR-in-arrears swap are exchanged for their values when the "assume forward rates are realized rule" is used.

Problem 32.8.

In the accrual swap discussed in the text, the fixed side accrues only when the floating reference rate lies below a certain level. Discuss how the analysis can be extended to cope with a situation where the fixed side accrues only when the floating reference rate is above one level and below another.

Suppose that the fixed rate accrues only when the floating reference rate is below R_X and above R_Y where $R_Y < R_X$. In this case the swap is a regular swap plus two series of binary options, one for each day of the life of the swap. Using the notation in the text, the risk-neutral probability that LIBOR will be above R_X on day i is $N(d_2)$ where

$$d_2 = \frac{\ln(F_i/R_X) - \sigma_i^2 t_i^2/2}{\sigma_i\sqrt{t_i}}$$

The probability that it will be below R_Y where $R_Y < R_X$ is $N(-d_2')$ where

$$d_2' = \frac{\ln(F_i/R_Y) - \sigma_i^2 t_i^2/2}{\sigma_i\sqrt{t_i}}$$

From the viewpoint of the party paying fixed, the swap is a regular swap plus binary options. The binary options corresponding to day i have a total value of

$$\frac{QL}{n_2}P(0, s_i)[N(d_2) + N(-d_2')]$$

(This ignores the small timing adjustment mentioned in Section 32.6.)

ASSIGNMENT QUESTIONS

Problem 32.9.

LIBOR zero rates are flat at 5% in the U.S and flat at 10% in Australia (both annually compounded). In a four-year swap Australian LIBOR is received and 9% is paid with both being applied to a USD principal of $10 million. Payments are exchanged annually. The volatility of all one-year forward rates in Australia is estimated to be 25%, the volatility of the forward USD–AUD exchange rate (AUD per USD) is 15% for all maturities, and the correlation between the two is 0.3. What is the value of the swap?

The fixed side consists of four payments of USD 0.9 million. The present value in millions of dollars is

$$\frac{0.9}{1.05} + \frac{0.9}{1.05^2} + \frac{0.9}{1.05^3} + \frac{0.9}{1.05^4} = 2.85$$

The forward Australian LIBOR rate is 10% with annual compounding. From Section 29.3 the quanto adjustment to the floating payment at time $t_i + 1$ is

$$0.1 \times 0.7 \times 0.15 \times 0.25 t_i = 0.002625 t_i$$

The value of the floating payments is therefore

$$\frac{1}{1.05} + \frac{1.02625}{1.05^2} + \frac{1.0525}{1.05^3} + \frac{1.07875}{1.05^4} = 3.28$$

The value of the swap is $3.28 - 2.85 = 0.43$.

Problem 32.10.

Estimate the interest rate paid by P&G on the 5/30 swap in Section 32.7 if a) the CP rate is 6.5% and the Treasury yield curve is flat at 6% and b) the CP rate is 7.5% and the Treasury yield curve is flat at 7% with semiannual compounding.

When the CP rate is 6.5% and Treasury rates are 6% with semiannual compounding, the CMT% is 6% and an Excel spreadsheet can be used to show that the price of a 30-year bond with a 6.25% coupon is about 103.46. The spread is zero and the rate paid by P&G is 5.75%. When the CP rate is 7.5% and Treasury rates are 7% with semiannual compounding, the CMT% is 7% and the price of a 30-year bond with a 6.25% coupon is about 90.65. The spread is therefore

$$\max[0, (98.5 \times 7/5.78 - 90.65)/100]$$

or 28.64%. The rate paid by P&G is 35.39%.

Problem 32.11.

Suppose that you are trading a LIBOR-in-arrears swap with an unsophisticated counterparty who does not make convexity adjustments. To take advantage of the situation, should you be paying fixed or receiving fixed? How should you try to structure the swap as far as its life and payment frequencies?

Consider the situation where the yield curve is flat at 10% per annum with annual compounding. All cap volatilities are 18%. Estimate the difference between the way a sophisticated trader and an unsophisticated trader would value a LIBOR-in-arrears swap where payments are made annually and the life of the swap is (a) 5 years, (b) 10 years, and (c) 20 years. Assume a notional principal of $1 million.

You should be paying fixed and receiving floating. The counterparty will value the floating payments less than you because it does not make a convexity adjustment increasing forward rates. The size of the convexity adjustment for a forward rate increases with the forward rate, the forward rate volatility, the time between resets, and the time until the forward rate is observed. We therefore maximize the impact of the convexity adjustment by choosing long swaps involving high interest rate currencies, where the interest rate volatility is high and there is a long time between resets.

The convexity adjustment for the payment at time t_i is

$$\frac{0.1^2 \times 0.18^2 \times 1 \times t_i}{1.1}$$

This is $0.000295t_i$. For a five year LIBOR-in-arrears swap the value of the convexity adjustment is

$$1,000,000 \sum_{i=1}^{5} \frac{0.000295i}{1.1^i}$$

or $3137.7. Similarly the value of the convexity adjustments for 10 and 20 year swaps is $8,552.4 and $18,827.5.

Problem 32.12.

Suppose that the LIBOR zero rate is flat at 5% with annual compounding. In a five-year swap, company X pays a fixed rate of 6% and receives LIBOR. The volatility of the two-year swap rate in three years is 20%.

a. *What is the value of the swap?*
b. *Use DerivaGem to calculate the value of the swap if company X has the option to cancel after three years.*
c. *Use DerivaGem to calculate the value of the swap if the counterparty has the option to cancel after three years.*
d. *What is the value of the swap if either side can cancel at the end of three years?*

(a) Because the LIBOR zero curve is flat at 5% with annual compounding, the five-year swap rate for an annual-pay swap is also 5%. (As explained in Chapter 7 swap rates are par yields.) A swap where 5% is paid and LIBOR is receive would therefore be worth zero. A swap where 6% is paid and LIBOR is received has the same value as an instrument that pays 1% per year. Its value in millions of dollars is therefore

$$-\frac{1}{1.05}-\frac{1}{1.05^2}-\frac{1}{1.05^3}-\frac{1}{1.05^4}-\frac{1}{1.05^5}=-4.33$$

(b) In this case company X has, in addition to the swap in (a), a European swap option to enter into a two-year swap in three years. The swap gives company X the right to receive 6% and pay LIBOR. We value this in DerivaGem by using the Caps and Swap Options worksheet. We choose Swap Option as the Underling Type, set the Principal to 100, the Settlement Frequency to Annual, the Swap Rate to 6%, and the Volatility to 20%. The Start (Years) is 3 and the End (Years) is 5. The Pricing Model is Black-European. We choose Rec Fixed and do not check the Imply Volatility or Imply Breakeven Rate boxes. All zero rates are 4.879% with continuous compounding. We therefore need only enter 4.879% for one maturity. The value of the swap option is given as 2.18. The value of the swap with the cancelation option is therefore

$$-4.33+2.18=-2.15$$

(c) In this case company X has, in addition to the swap in (a), granted an option to the counterparty. The option gives the counterparty the right to pay 6% and receive LIBOR on a two-year swap in three years. We can value this in DerivaGem using the same inputs as in (b) but with the Pay Fixed instead of the Rec Fixed being chosen. The value of the swap option is 0.57. The value of the swap to company X is

$$-4.33-0.57=-4.90$$

(d) In this case company X is long the Rec Fixed option and short the Pay Fixed option. The value of the swap is therefore

$$-4.33+2.18-0.57=-2.72$$

It is certain that one of the two sides will exercise its option to cancel in three years. The swap is therefore to all intents and purposes a three-year swap with no embedded options. Its value can also be calculated as

$$-\frac{1}{1.05}-\frac{1}{1.05^2}-\frac{1}{1.05^3}=-2.72$$

CHAPTER 33
Real Options

Notes for the Instructor

The real options approach to capital investment appraisal has become so popular in recent years that it is clearly appropriate to include a chapter on it in a derivatives text. It is potentially much easier to value embedded options, such as expansion and abandonment options, using the real options approach than using traditional approaches. (Some people argue that real options including the whole of options pricing should be a central part of any course in corporate finance!)

I use two examples to illustrate the real options approach. The first is the Schwartz and Moon study aimed at valuing Amazon.com. The second is an oil exploration example that I developed myself. I find that students relate well to both examples. The oil exploration example also serves as a vehicle to illustrate how a trinomial tree can be built for a commodity price.

Both Problem 33.8 and 33.9 work well as assignment questions.

QUESTIONS AND PROBLEMS

Problem 33.1.

Explain the difference between the net present value approach and the risk-neutral valuation approach for valuing a new capital investment opportunity. What are the advantages of the risk-neutral valuation approach for valuing real options?

In the net present value approach, cash flows are estimated in the real world and discounted at a risk-adjusted discount rate. In the risk-neutral valuation approach, cash flows are estimated in the risk-neutral world and discounted at the risk-free interest rate. The risk-neutral valuation approach is arguably more appropriate for valuing real options because it is very difficult to determine the appropriate risk-adjusted discount rate when options are valued.

Problem 33.2.

The market price of risk for copper is 0.5, the volatility of copper prices is 20% per annum, the spot price is 80 cents per pound, and the six-month futures price is 75 cents per pound. What is the expected percentage growth rate in copper prices over the next six months?

In a risk-neutral world the expected price of copper in six months is 75 cents. This corresponds to an expected growth rate of $2\ln(75/80) = -12.9\%$ per annum. The decrease in the growth rate when we move from the real world to the risk-neutral world is the volatility of copper times its market price of risk. This is $0.2 \times 0.5 = 0.1$ or 10% per annum. It follows that the expected growth rate of the price of copper in the real world is −2.9%.

Problem 33.3.

Consider a commodity with constant volatility, σ. and an expected growth rate that is a function solely of time. Show that in the traditional risk-neutral world,

$$\ln S_T \sim \phi\left[\ln F(T) - \frac{\sigma^2}{2}T,\ \sigma^2 T\right]$$

where S_T is the value of the commodity at time T, $F(t)$ is the futures price at time zero for a contract maturing at time t, and $\phi(m, v)$ is a normal distribution with mean m and variance v.

In this case

$$\frac{dS}{S} = \mu(t)\, dt + \sigma\, dz$$

or

$$d \ln S = [\mu(t) - \sigma^2/2]\, dt + \sigma\, dz$$

so that $\ln S_T$ is normal with mean

$$\ln S_0 + \int_{t=0}^{T} \mu(t)dt - \sigma^2 T/2$$

and standard deviation $\sigma\sqrt{T}$. Section 33.5 shows that

$$\mu(t) = \frac{\partial}{\partial t}[\ln F(t)]$$

so that

$$\int_{t=0}^{T} \mu(t)dt = \ln F(T) - \ln F(0)$$

Since $F(0) = S_0$ the result follows.

Problem 33.4.

Derive a relationship between the convenience yield of a commodity and its market price of risk.

We explained the concept of a convenience yield for a commodity in Chapter 5. It is a measure of the benefits realized from ownership of the physical commodity that are not realized by the holders of a futures contract. If y is the convenience yield and u is the storage cost, equation (5.17) shows that the commodity behaves like an investment asset that provides a return equal to $y - u$. In a risk-neutral world its growth is, therefore,

$$r - (y - u) = r - y + u$$

The convenience yield of a commodity can be related to its market price of risk. From Section 33.2, the expected growth of the commodity price in a risk-neutral world is $m - \lambda s$, where m is its expected growth in the real world, s its volatility, and λ is its market price of risk. It follows that

$$m - \lambda s = r - y + u$$

or

$$y = r + u - m + \lambda s$$

Problem 33.5.

The correlation between a company's gross revenue and the market index is 0.2. The excess return of the market over the risk-free rate is 6% and the volatility of the market index is 18%. What is the market price of risk for the company's revenue?

In equation (33.2) $\rho = 0.2$, $\mu_m - r = 0.06$, and $\sigma_m = 0.18$. It follows that the market price of risk lambda is

$$\frac{0.2 \times 0.06}{0.18} = 0.067$$

Problem 33.6.

A company can buy an option for the delivery of one million units of a commodity in three years at \$25 per unit. The three year futures price is \$24. The risk-free interest rate is 5% per annum with continuous compounding and the volatility of the futures price is 20% per annum. How much is the option worth?

The option can be valued using Black's model. In this case $F_0 = 24$, $K = 25$, $r = 0.05$, $\sigma = 0.2$, and $T = 3$. The value of a option to purchase one unit at \$25 is

$$e^{-rT}[F_0 N(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}}$$

This is 2.489. The value of the option to purchase one million units is therefore \$2,489,000.

Problem 33.7.

A driver entering into a car lease agreement can obtain the right to buy the car in four years for \$10,000. The current value of the car is \$30,000. The value of the car, S, is expected to follow the process

$$dS = \mu S\,dt + \sigma S\,dz$$

where $\mu = -0.25$, $\sigma = 0.15$, and dz is a Wiener process. The market price of risk for the car price is estimated to be -0.1. What is the value of the option? Assume that the risk-free rate for all maturities is 6%.

The expected growth rate of the car price in a risk-neutral world is $-0.25 - (-0.1 \times 0.15) = -0.235$ The expected value of the car in a risk-neutral world in four years, $\hat{E}(S_T)$, is therefore $30,000e^{-0.235\times 4} = \$11,719$. Using the result in the appendix to Chapter 13 the value of the option is

$$e^{-rT}[\hat{E}(S_T)N(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{\ln(\hat{E}(S_T)/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(\hat{E}(S_T)/K) - \sigma^2 T/2}{\sigma\sqrt{T}}$$

$r = 0.06$, $\sigma = 0.15$, $T = 4$, and $K = 10,000$. It is \$1,832.

ASSIGNMENT QUESTIONS

Problem 33.8.

Suppose that the spot price, 6-month futures price, and 12-month futures price for wheat are 250, 260, and 270 cents per bushel, respectively. Suppose that the price of wheat follows the process in equation (33.4) with $a = 0.05$ and $\sigma = 0.15$. Construct a two-time-step tree for the price of wheat in a risk-neutral world.

A farmer has a project that involves an expenditure of \$10,000 and a further expenditure of \$90,000 in six months. It will increase wheat that is harvested and sold by 40,000 bushels in one year. What is the value of the project? Suppose that the farmer can abandon the project in six months and avoid paying the \$90,000 cost at that time. What is the value of the abandonment option? Assume a risk-free rate of 5% with continuous compounding.

In this case $a = 0.05$ and $\sigma = 0.15$. We first define a variable X that follows the process

$$dX = -a\,dt + \sigma\,dz$$

A tree for X constructed in the way described in Chapter 28 is shown in Figure M33.1. We now displace nodes so that the tree models $\ln S$ in a risk-neutral world where S is the price of wheat. The displacements are chosen so that the initial price of wheat is 250 cents and the expected prices at the ends of the first and second time steps are 260 and 270 cents, respectively. Suppose that the displacement to give $\ln S$ at the end of the first time step is α_1. Then

$$0.1667e^{\alpha_1+0.1837} + 0.6666e^{\alpha_1} + 0.1667e^{\alpha_1-0.1837} = 260$$

so that $\alpha_1 = 5.5551$ The probabilities of nodes E, F, G, H, and I being reached are 0.0257, 0.2221, 0.5043, 0.2221, and 0.0257, respectively. Suppose that the displacement to give $\ln S$ at the end of the second step is α_2. Then

$$0.0257e^{\alpha_2+0.3674} + 0.2221e^{\alpha_2+0.1837} + 0.5043e^{\alpha_2} + 0.2221e^{\alpha_2-0.1837}$$

$$+0.0257e^{\alpha_2-0.3674}=270$$

so that $\alpha_2=5.5874$. This leads to the tree for the price of wheat shown in Figure M33.2.

Using risk-neutral valuation the value of the project (in thousands of dollars) is

$$-10-90e^{-0.05\times0.5}+2.70\times40e^{-0.05\times1}=4.94$$

This shows that the project is worth undertaking. Figure M33.3 shows the value of the project on a tree. The project should be abandoned at node D for a saving of 2.41. Figure M33.4 shows that the abandonment option is worth 0.39.

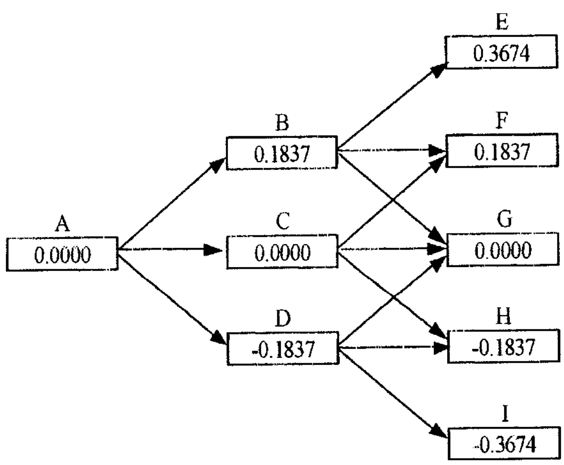
Problem 33.9.

In the example considered in Section 33.6

a. *What is the value of the abandonment option if it costs \$3 million rather than zero?*

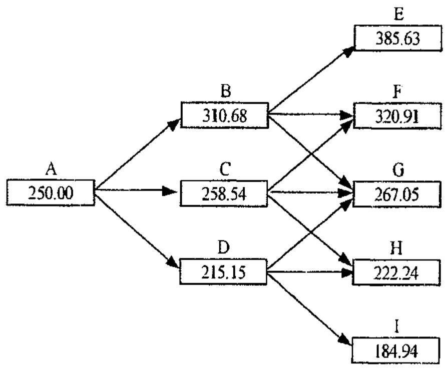
b. *What is the value of the expansion option if it costs \$5 million rather than \$2 million?*

Figure M33.5 shows what Figure 33.4 in the text becomes if the abandonment option costs \$3 million. The value of the abandonment option reduces from 1.94 to 1.21. Similarly Figure M33.6 below shows what Figure 33.5 in the text becomes if the expansion option costs \$5 million. The value of the expansion option reduces from 1.06 to 0.40.



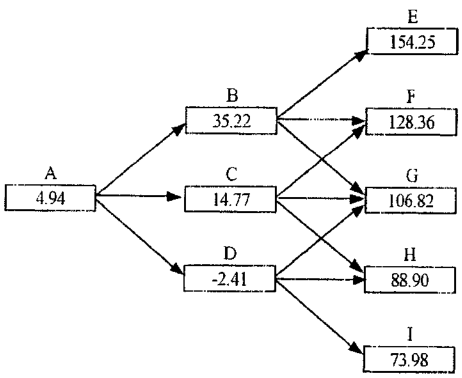
Node	A	B	C	D
p_u	0.1667	0.1545	0.1667	0.1795
p_m	0.6666	0.6660	0.6666	0.6660
p_d	0.1667	0.1795	0.1667	0.1545

Figure M33.1 Tree for X in Problem 33.8



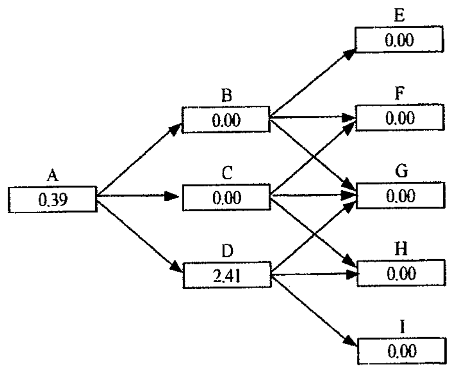
Node	A	B	C	D
p_u	0 1667	0.1545	0.1667	0 1795
p_m	0.6666	0.6660	0.6666	0.6660
p_d	0.1667	0.1795	0.1667	0.1545

Figure M33.2 Tree for price of wheat in Problem 33.8



Node	A	B	C	D
p_u	0.1667	0 1545	0 1667	0.1795
p_m	0.6666	0.6660	0 6666	0.6660
p_d	0.1667	0.1795	0.1667	0.1545

Figure M33.3 Tree for value of project in Problem 33.8



Node	A	B	C	D
p_u	0.1667	0.1545	0.1667	0.1795
p_m	0.6666	0.6660	0.6666	0.6660
p_d	0.1667	0.1795	0.1667	0.1545

Figure M33.4 Tree for abandonment option in Problem 33.8

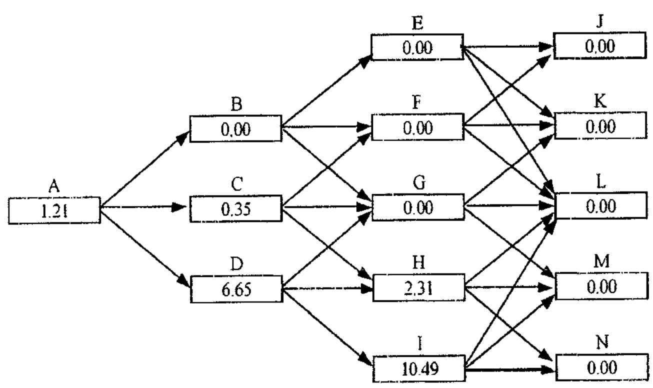


Figure M33.5 Tree for abandonment option in Problem 33.9

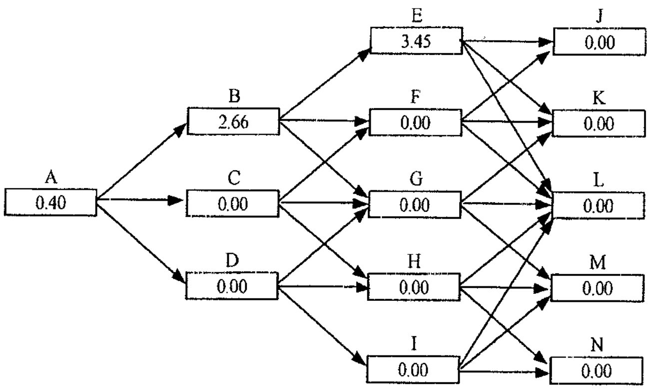


Figure M33.6 Tree for expansion option in Problem 33.9

CHAPTER 34

Derivatives Mishaps and What We Can Learn from Them

Notes for the Instructor

This chapter describes the well-publicized derivatives disasters of the last 20 years and discusses the lessons that can be learned from them.

Chapter 32 is a great chapter for the final class of a course. Students love talking about the derivatives disasters of the past, what went wrong, why it went wrong, etc. I find that questions about some of the disasters (particularly LTCM) often arise relatively early in my courses. It is useful to be able to say that we are going to talk about them in some detail towards the end of the semester.

I like to go through the lessons one by one. When a derivatives disaster is relevant to a particular lesson, we talk about it in some detail. It not a good idea to sound too self righteous when discussing the disasters. It is a good idea to say things like "Of course it is easy to be wise after the event." Asking students whether they think the derivatives business has learned its lessons or whether there will be more disasters in the future can generate interesting viewpoints.

I like to try to end on a positive note such as: "The derivatives industry is huge and it is here to stay. These disasters are not representative of what happens in the industry. They constitute a very very small proportion of the total trades. Most trades are entered into by corporations for sensible hedging and risk management purposes. But the disasters are interesting because of what we can learn from them."

Test Bank Questions

Pages 420 to 446 contain test bank questions for chapters 1 to 21. Some questions are multiple choice; others have numerical answers that can easily be computed using a calculator. In each of the 21 tests, there are a total of ten answers that can be quickly graded as correct or incorrect.

The questions can be used at the beginning of a class to provide a quick test of whether students understand the material covered in the previous class. Alternatively, they can be used to provide instructors with ideas for midterm or final exam questions.

The answers are on pages 447–448.

Microsoft Word files for the tests can be downloaded from the Pearson Instructor Resource Center.